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Plancherel–Rotach-Type Asymptotics for Orthogonal Polynomials Associated with $\exp(-x^6/6)$ *

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All orthogonal polynomial systems satisfy a recurrence formula:

$$xp_n(x) = a_{n+1} p_{n+1}(x) + b_n p_n(x) + a_n p_{n-1}(x). \tag{1}$$

In case the weight function (over $(-\infty, \infty)$) is $\exp(-x^6/6)$, $b_n = 0$, and the a_n also satisfy the recurrence formula: $n = a_n^2(a_{n+2}^2 a_{n+1}^2 + a_{n+1}^4 + 2a_{n+1}^2 a_n^2 + a_{n+1}^2 a_{n-1}^2 + a_n^4 + 2a_n^2 a_{n-1}^2 + a_{n-1}^4 + a_{n-2}^2 a_{n-1}^2)$. The existence of an asymptotic series for a_n has been recently proved by A. Máté and P. G. Nevai. The present paper is based on the above two recurrence formulas. First, we apply Shohat's method to (1) to obtain a differential equation for $P_n(x)$. Then, by applying the asymptotics of a_n to this differential equation, we obtain Plancherel–Rotach-type asymptotics for $P_n(x)$ in the interval $|x| \leqslant cx_n$, where x_n denotes the largest zero of $P_n(x)$, and $0 < c < 1$.

1. INTRODUCTION

Let $w(x) = \exp(-x^6/6)$, $x \in R$, and let $P_n(x) = \gamma_n x^n + \gamma_{n-1} x^{n-1} + \cdots + \gamma_0$, $\gamma_n > 0$ denote the orthonormalized polynomials corresponding to $w(x)$. Then $\{P_n(x)\}$ satisfies the recursion formula

$$xp_n(x) = a_{n+1} p_{n+1}(x) + a_n p_{n-1}(x), \tag{1.1}$$

where $a_n = \gamma_{n-1}/\gamma_n$. G. Freud [2] showed that the sequence $\{a_n\}$ also satisfies the recursion formula

$$\begin{aligned} n = a_n^2[&a_{n+2}^2 a_{n+1}^2 + a_{n+1}^4 + 2a_{n+1}^2 a_n^2 + a_{n+1}^2 a_{n-1}^2 + a_n^4 + 2a_n^2 a_{n-1}^2 \\ &+ a_{n-1}^4 + a_{n-1}^2 a_{n-2}^2], \qquad n = 1, 2, \ldots, a_0 = 0, a_{-1} = 0. \end{aligned} \tag{1.2}$$

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Recently, A. Máté and P. G. Nevai [6] proved that there exists an asymptotic series for a_n; in particular,

$$a_n = \left(\frac{n}{10}\right)^{1/6} \left[1 + \frac{1}{36}\frac{1}{n^2} + O\left(\frac{1}{n^4}\right)\right]. \tag{1.3}$$

All the classical orthogonal polynomials satisfy a simple second order linear homogeneous differential equation with known initial conditions. Applying the Liouville–Steklov method [13] to these equations, we can immediately find asymptotics for these polynomials. However, there exist other orthogonal polynomial systems which do not satisfy a simple or explicit differential equation. Indeed, finding the associated differential equation requires a great deal of effort; for example, in the case $w(x) = \exp(-x^4)$, P. G. Nevai [8] proved that the corresponding orthogonal polynomials $P_n(x)$ satisfy the differential equation:

$$P'_n(x) - \left(4x^3 + \frac{2x}{\phi_n(x)}\right) p'_n(x)$$
$$+ 4a_n^2 \left(4\phi_n(x)\, \phi_{n-1}(x) + 1 - 4a_n^2 x^2 - 4x^4 - 2\frac{x^2}{\phi_n(x)}\right) p_n(x) = 0$$

where $\phi_n(x) = a_{n+1}^2 + a_n^2 + x^2$ and a_n are the coefficients in the recursion formula for $p_n(x)$. The same idea was also used by Shohat [12] to find the differential equation of $p_n(x)$ associated to the weight function $w(x) = A^{-1} \exp(\int BA^{-1}\, dx)$, where A and B are certain fixed polynomials. In this paper, we will use the same method to find a differential equation of $p_n(x)$ associated to the weight function $\exp(-x^6/6)$. Then we will apply P. G. Nevai's technique [9] to find asymptotics for the polynomial $P_n(x)$ which is valid uniformly for

$$|x| \leqslant c\left(\frac{32n}{5}\right)^{1/6}, \qquad \text{where} \quad 0 < c < 1. \tag{1.4}$$

G. Freud [3] showed that $(32n/5)^{1/6}$ is asymptotically equivalent to the greatest zero, denoted as x_n, of $p_n(x)$. In view of (1.4), the asymptotic for $p_n(x)$ will be valid uniformly for $|x| \leqslant cx_n$, where $0 < c < 1$. Additionally, we will also get an asymptotic for $\sum_{k=0}^{n-1} p_k^2(x) \exp(-x^6/6)$. These reults are the extensions of three theorems in P. G. Nevai's paper [9], which state:

THEOREM A. *Let* $p_n(x)$ *be orthonormal polynomials associated with* $\exp(-x^4)$, $0 < \varepsilon < \pi/2$ *be fixed and* $x = (4n/3)^{1/4} \cos\theta$. *Then the asymptotic formula*

$$p_n(x)\exp(-x^4/2) = 12^{1/8}\pi^{-1/2}n^{-1/8}(\sin\theta)^{-1/2} \cdot \cos\left[n\cdot 12^{-1}(12\theta - 4\sin 2\theta - \sin 4\theta) + \frac{\theta}{2} + \frac{\pi}{4}\right] + O(n^{-9/8})$$

holds uniformly for $n = 1, 2, \ldots,$ *and* $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$.

THEOREM B. *Let* $\{p_n(x)\}$ *be orthonormal polynomials associated with* $\exp(-x^4)$, $0 < \varepsilon < \pi/2$ *be fixed and* $x = (4n/3)^{1/4}\cos\theta$. *Then the asymptotic formula*

$$n^{-3/4}\sum_{k=0}^{n-1} p_k^2(x)\exp(-x^4) = 2(12)^{1/4}(3\pi)^{-1}\sin\theta(1 + 2\cos^2\theta) + O(n^{-1})$$

holds uniformly for $n = 1, 2, \ldots,$ *and* $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$.

An application of Theorem A was given in P. G. Nevai's paper [11], which states:

THEOREM C. *For a given interval* Δ,

$$\begin{aligned} p_n(x)&\exp(-x^4/2) \\ &= 12^{1/8}\pi^{-1/2}n^{-1/8}\cdot\Bigg\{(1 + B_1(x)\,n^{-1/2}) \\ &\times\cos\left[\left(\frac{64}{27}\right)^{1/4} xn^{3/4} + \left(\frac{1}{12}\right)^{1/4} x^3 n^{1/4} - \frac{n\pi}{2}\right] \\ &+ (B_2(x)n^{-1/4} + B_3(x)n^{-3/4})\sin\left[\left(\frac{64}{27}\right)^{1/4} xn^{3/4}\right. \\ &\left.+\left(\frac{1}{12}\right)^{1/4} x^3 n^{1/4} - \frac{n\pi}{2}\right]\Bigg\} + O(n^{-9/8}) \end{aligned}$$

holds uniformly for $n = 1, 2, \ldots$ *and* $x \in \Delta$, *where*

$$B_1(x) = \frac{1}{8}\left(\frac{3}{4}\right)^{1/2}\left(x^2 + \frac{9}{10}x^6 - \frac{81}{400}x^{10}\right),$$

$$B_2(x) = \frac{1}{2}\left(\frac{3}{4}\right)^{1/4}\left(-x + \frac{9}{20}x^5\right),$$

and

$$B_3(x) = \frac{1}{4}\left(\frac{3}{4}\right)^{3/4}\left(-\frac{3}{4}x^3 + \frac{163}{560}x^7 + \frac{81}{1600}x^{11} - \frac{243}{32000}x^{15}\right).$$

2. Results

THEOREM 1. *Let* $0<\varepsilon<\pi/2$ *be fixed and let* $x=(32n/5)^{1/6}\cos\theta$. *Then the asymptotic formula*

$$\begin{aligned}
&p_n(x)\exp(-x^6/12)\\
&\quad=10^{1/12}\pi^{-1/2}n^{-1/12}(\sin\theta)^{-1/2}\\
&\qquad\times\cos\left[\frac{n}{60}(60\theta-15\sin 2\theta-6\sin 4\theta-\sin 6\theta)+\frac{\theta}{2}-\frac{\pi}{4}\right]+O(n^{-13/12})
\end{aligned}$$

holds uniformly for $n=1, 2,\ldots$ *and* $\varepsilon\leqslant\theta\leqslant\pi-\varepsilon$.

THEOREM 2. *If* $0<\varepsilon<\pi/2$ *is fixed and* $x=(32n/5)^{1/6}\cos\theta$, *then*

$$\begin{aligned}
&n^{-5/6}\sum_{k=0}^{n-1}p_k^2(x)\exp(-x^6/6)\\
&\qquad=10^{-5/6}\pi^{-1}\sin\theta\cdot(16\cos^4\theta+8\cos^2\theta+6)+O\left(\frac{1}{n}\right)
\end{aligned}$$

holds uniformly for $n=1, 2, 3,\ldots$ *and* $\varepsilon\leqslant\theta\leqslant\pi-\varepsilon$.

THEOREM 3. *For a given interval* Δ,

$$\begin{aligned}
&p_n(x)\exp(-x^6/12)\\
&\quad=10^{1/12}\pi^{-1/2}n^{-1/12}\cdot[(1+B_1(x)n^{-1/3}+B_2(x)n^{-2/3})\\
&\qquad\cdot\cos F+(B_3(x)n^{-1/6}+B_4(x)n^{-1/2}+B_5(x)n^{-5/6})\cdot\sin F]\\
&\quad+O(n^{-13/12})
\end{aligned}$$

holds uniformly for $n=1, 2, 3,\ldots$ *and* $x\in\Delta$, *where*

$$F=6\left(\frac{n}{10}\right)^{5/6}x+\frac{5}{12}\left(\frac{n}{10}\right)^{1/2}x^3+\frac{9}{64}\left(\frac{n}{10}\right)^{1/6}x^5-\frac{n\pi}{2},$$

$$B_1(x)=\frac{-225}{6272}\left(\frac{5}{32}\right)^{7/3}x^{14}+\frac{15}{112}\left(\frac{5}{32}\right)^{4/3}x^8+\frac{1}{8}\left(\frac{5}{32}\right)^{1/3}x^2,$$

$$\begin{aligned}
B_2(x)=&\frac{16875}{78675968}\left(\frac{5}{32}\right)^{14/3}x^{28}-\frac{1125}{702464}\left(\frac{5}{32}\right)^{11/3}x^{22}-\frac{3125}{150528}\left(\frac{5}{32}\right)^{8/3}x^{16}\\
&+\frac{325}{4032}\left(\frac{5}{32}\right)^{5/3}x^{10}+\frac{11}{128}\left(\frac{5}{32}\right)^{2/3}x^4,
\end{aligned}$$

$$B_3(x)=\frac{15}{56}\left(\frac{5}{32}\right)^{7/6}x^7-\frac{1}{2}\left(\frac{5}{32}\right)^{1/6}x,$$

$$B_4(x)=\frac{-1125}{351232}\left(\frac{5}{32}\right)^{7/2}x^{21}+\frac{225}{12544}\left(\frac{5}{32}\right)^{5/2}x^{15}$$
$$+\frac{95}{1008}\left(\frac{5}{32}\right)^{3/2}x^9-\frac{3}{16}\left(\frac{5}{32}\right)^{1/2}x^3,$$

and

$$B_5(x)=\frac{50625}{4405854208}\left(\frac{5}{32}\right)^{35/6}x^{35}-\frac{16875}{157351936}\left(\frac{5}{32}\right)^{29/6}x^{29}$$
$$-\frac{3625}{1404928}\left(\frac{5}{32}\right)^{23/6}x^{23}+\frac{4475}{301056}\left(\frac{5}{32}\right)^{17/6}x^{17}$$
$$+\frac{203881}{3548160}\left(\frac{5}{32}\right)^{11/6}x^{11}-\frac{31}{256}\left(\frac{5}{32}\right)^{5/6}x^5.$$

3. Lemmas

Lemma 1. *The orthonormal polynomials $\{p_n(x)\}$ associated to the weight function $w(x)=\exp(-x^6/6)$ satisfy the following equation*:

$$p_n'(x)=\frac{n}{a_n}p_{n-1}(x)+a_na_{n-1}a_{n-2}(a_{n+1}^2+a_n^2+a_{n-1}^2+a_{n-2}^2+a_{n-3}^2)\,p_{n-3}(x)$$
$$+a_na_{n-1}a_{n-2}a_{n-3}a_{n-4}p_{n-5}(x),$$

where $a_n=\gamma_{n-1}/\gamma_n$, $\gamma_n>0$ denotes the leading coefficient of $p_n(x)$.

Proof of Lemma 1. Expanding $p_n'(x)$ into Fourier series in $\{p_k(x)\}$, we obtain

$$p_n'(x)=\sum_{k=0}^{n-1}c_k\,p_k(x). \tag{3.1}$$

Multiplying (3.1) by $p_k(x)$ on both sides and then integrate w.r.t. $w(x)$, we have

$$c_k=\int_{-\infty}^{\infty}p_n'(x)\,p_k(x)\,w(x)\,dx$$
$$=\int_{-\infty}^{\infty}p_k(x)\,w(x)\,dp_n(x)$$

$$= [p_k(x)\, w(x)\, p_n(x)]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} p_n(x)[p_k(x)\, w(x)]'\, dx$$

$$= -\int_{-\infty}^{\infty} p_n(x)\, p_n'(x)\, w(x)\, dx + \int_{-\infty}^{\infty} p_n(x)\, p_k(x)\, x^5 w(x)\, dx. \quad (3.2)$$

Since the degree of $p_k'(x)$ is less than that of $p_n(x)$, by the orthogonality, the first term of the right-hand side in (3.2) vanishes. So

$$c_k = \int_{-\infty}^{\infty} p_n(x)\, p_k(x)\, x^5 w(x)\, dx. \quad (3.3)$$

From (3.3) and orthogonality, we see that if $k+5<n$ then $c_k=0$, in other words if $k<n-5$ then $c_k=0$; also from (3.2), if $n-1+k$ is odd then $c_k=0$, since $w(x)$ is an even function (see [13, p. 29]). This implies $c_{n-4}=c_{n-2}=0$. Therefore (3.1) becomes

$$p_n'(x) = c_{n-5}\, p_{n-5}(x) + c_{n-3}\, p_{n-3}(x) + c_{n-1}\, p_{n-1}(x). \quad (3.4)$$

Now we are going to expand $x^5 p_n(x)$ and then use (3.3) to find c_{n-5}, c_{n-3}, and c_{n-1}.

By repeated application on the recursion formula

$$x p_n(x) = a_{n+1}\, p_{n+1}(x) + a_n\, p_{n-1}(x), \quad (3.5)$$

we obtain

$$\begin{aligned} x^2 p_n(x) &= x \cdot a_{n+1}\, p_{n+1}(x) + x \cdot a_n\, p_{n-1}(x) \\ &= a_{n+1}[a_{n+2}\, p_{n+2}(x) + a_{n+1}\, p_n(x)] + a_n[a_n p_n(x) + a_{n-1}\, p_{n-2}(x)] \\ &= a_{n+1} a_{n+2}\, p_{n+2}(x) + (a_{n+1}^2 + a_n^2)\, p_n(x) + a_n a_{n-1}\, p_{n-2}(x). \quad (3.6) \end{aligned}$$

Using (3.6), we see that

$$\begin{aligned} x^3 p_n(x) &= x \cdot x^2 p_n(x) \\ &= (a_{n+1} a_{n+2} a_{n+3})\, p_{n+3}(x) \\ &\quad + (a_{n+1} a_{n+2}^2 + a_{n+1}^3 + a_n^2 a_{n+1})\, p_{n+1}(x) \\ &\quad + (a_{n+1}^2 a_n + a_n^3 + a_n a_{n-1}^2)\, p_{n-1}(x) + (a_n a_{n-1} a_{n-2})\, p_{n-3}(x). \end{aligned}$$

Following the above procedure, we also have

$$\begin{aligned} x^4 p_n(x) &= (a_{n+1} a_{n+2} a_{n+3} a_{n+4})\, p_{n+4}(x) \\ &\quad + (a_{n+1} a_{n+2} a_{n+3}^2 + a_{n+1} a_{n+2}^3 + a_{n+1}^3 a_{n+2} \\ &\quad + a_n^2 a_{n+1} a_{n+2})\, p_{n+2}(x) \end{aligned}$$

$$+ (a_{n+1}^2 a_{n+2}^2 + a_{n+1}^4 + a_n^2 a_{n+1}^2 + a_{n+1}^2 a_n^2 + a_n^4 + a_n^2 a_{n-1}^2)\, p_n(x)$$
$$+ (a_{n+1}^2 a_n a_{n-1} + a_n a_{n-1}^3 + a_{n-1} a_n^3 + a_n a_{n-1} a_{n-2}^2)\, p_{n-2}(x)$$
$$+ (a_n a_{n-1} a_{n-2} a_{n-3})\, p_{n-4}(x).$$

and

$$\begin{aligned} x^5 p_n(x) = {} & (a_{n+1} a_{n+2} a_{n+3} a_{n+4} a_{n+5})\, p_{n+5}(x) + (a_{n+1} a_{n+2} a_{n+3} a_{n+4}^2 \\ & + a_{n+1} a_{n+2} a_{n+3}^3 + a_{n+1} a_{n+2}^3 a_{n+3} + a_{n+1}^3 a_{n+2} a_{n+3} \\ & + a_n^2 a_{n+1} a_{n+2} a_{n+3})\, p_{n+3}(x) + (a_{n+3}^2 a_{n+2}^2 a_{n+1} + a_{n+2}^4 a_{n+1} \\ & + a_{n+2}^2 a_{n+1}^3 + a_{n+2}^2 a_n^2 a_{n+1} + a_{n+1}^3 a_{n+2}^2 + a_{n+1}^5 + 2a_{n+1}^3 a_n^2 \\ & + a_{n+1} a_n^4 + a_{n+1} a_n^2 a_{n-1}^2)\, P_{n+1}(x) + (a_n a_{n+1}^2 a_{n+2}^2 + a_n a_{n+1}^4 \\ & + 2a_n^3 a_{n+1}^2 + a_n^5 + a_n^3 a_{n-1}^2 + a_{n-1}^2 a_n a_{n+1}^2 + a_{n-1}^2 a_n^3 + a_{n-1}^4 a_n \\ & + a_{n-1}^2 a_{n-2}^2 a_n)\, p_{n-1}(x) + (a_{n+1}^2 a_n a_{n-1} a_{n-2} + a_n^3 a_{n-1} a_{n-2} \\ & + a_n a_{n-1}^3 a_{n-2} + a_n a_{n-1} a_{n-2}^3 + a_n a_{n-1} a_{n-2} a_{n-3}^2)\, p_{n-3}(x) \\ & + a_n a_{n-1} a_{n-2} a_{n-3} a_{n-4}\, p_{n-5}(x). \end{aligned} \tag{3.7}$$

Multiplying (3.7) by $p_{n-1}(x)\, w(x)$ on both sides, then integrate and take (3.3) into account, we obtain

$$\begin{aligned} c_{n-1} = {} & a_n(a_{n+2}^2 a_{n+1}^2 + a_{n+1}^4 + 2a_{n+1}^2 a_n^2 + a_{n+1}^2 a_{n-1}^2 \\ & + a_n^4 + 2a_n^2 a_{n-1}^2 + a_{n-1}^4 + a_{n-1}^2 a_{n-2}^2). \end{aligned} \tag{3.8}$$

In view of (1.2), the right-hand side of (3.8) is exactly equal to n/a_n. Therefore,

$$c_{n-1} = \frac{n}{a_n}. \tag{3.9}$$

Similarly,

$$c_{n-3} = a_n a_{n-1} a_{n-2}(a_{n+1}^2 + a_n^2 + a_{n-1}^2 + a_{n-2}^2 + a_{n-3}^2), \tag{3.10}$$

and

$$c_{n-5} = a_n a_{n-1} a_{n-2} a_{n-3} a_{n-4}. \tag{3.11}$$

The lemma now follows from (3.4), (3.9), (3.10) and (3.11). In the following lemma, we will rewrite $p_n'(x)$ in terms of $p_{n-1}(x)$ and $p_n(x)$.

LEMMA 2. *If* $w(x)=\exp(-x^6/6)$, *then the corresponding polynomials* $\{p_n(x)\}$ *satisfy the following two equations*:

$$p_n'(x)=a_n\phi_n(x)\,p_{n-1}(x)-\pi_n(x)\,p_n(x), \tag{3.12}$$

and

$$\begin{aligned} p_n''(x)=&[2a_n(a_{n+1}^2+a_n^2)x+4a_nx^3+a_nx^5\phi_n(x)]\,p_{n-1}(x)\\ &+[\pi_n^2(x)-\pi_n'(x)-a_n^2\phi_n(x)\,\phi_{n-1}(x)]\,p_n(x), \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \phi_n(x)=&a_{n+1}^2(a_{n+2}^2+a_{n+1}^2+a_n^2)+a_n^2(a_{n+1}^2+a_n^2+a_{n-1}^2)\\ &+x^2(a_{n+1}^2+a_n^2+x^2), \end{aligned} \tag{3.14}$$

and

$$\pi_n(x)=a_n^2x(a_n^2+a_{n+1}^2+a_{n-1}^2+x^2). \tag{3.15}$$

Proof of Lemma 2. First of all, we will express $p_{n-5}(x)$ and $p_{n-3}(x)$ in terms of $p_n(x)$ and $p_{n-1}(x)$, and then apply Lemma 1. From the recursion formula $xp_{n-1}(x)=a_n\,p_n(x)+a_{n-1}\,p_{n-2}(x)$, we have

$$p_{n-2}(x)=\frac{1}{a_{n-1}}\,[xp_{n-1}(x)-a_n\,p_n(x)]. \tag{3.16}$$

This implies

$$p_{n-3}(x)=\frac{1}{a_{n-2}}\,[xp_{n-2}(x)-a_{n-1}\,p_{n-1}(x)].$$

Applying (3.16), we obtain

$$\begin{aligned} p_{n-3}(x)&=\frac{x}{a_{n-2}}\left[\frac{1}{a_{n-1}}\,(xp_{n-1}(x)-a_n\,p_n(x))\right]-\frac{a^{n-1}}{a_{n-2}}\,p_{n-1}(x)\\ &=\left[\frac{x^2}{a_{n-1}a_{n-2}}-\frac{a_{n-1}}{a_{n-2}}\right]p_{n-1}(x)-\frac{a_nx}{a_{n-2}a_{n-1}}\,p_n(x). \end{aligned} \tag{3.17}$$

If we substitute $n-2$ for n into (3.17), then we get

$$p_{n-5}(x)=\left(\frac{x^2}{a_{n-4}a_{n-3}}-\frac{a_{n-3}}{a_{n-4}}\right)p_{n-3}(x)-\frac{a_{n-2}x}{a_{n-4}a_{n-3}}\,p_{n-2}(x). \tag{3.18}$$

Applying (3.16) and (3.17) to the right-hand side of (3.18), we obtain

$$\begin{aligned}
p_{n-5}(x) &= \left(\frac{x^2}{a_{n-4}a_{n-3}} - \frac{a_{n-3}}{a_{n-4}}\right)\left[\left(\frac{x^2}{a_{n-1}a_{n-2}} - \frac{a_{n-1}}{a_{n-2}}\right) p_{n-1}(x)\right. \\
&\quad \left. - \frac{a_n x}{a_{n-2}a_{n-1}}\, p_n(x)\right] - \frac{a_{n-2}x}{a_{n-4}a_{n-3}}\left[\frac{x p_{n-1}(x)}{a_{n-1}} - \frac{a_n p_n(x)}{a_{n-1}}\right] \\
&= \left(\frac{a_n a_{n-2} x}{a_{n-4}a_{n-3}a_{n-1}} + \frac{a_{n-3}a_n x}{a_{n-4}a_{n-2}a_{n-1}} - \frac{a_n x^3}{a_{n-4}a_{n-3}a_{n-2}a_{n-1}}\right) p_n(x) \\
&\quad + \left(\frac{x^4}{a_{n-4}a_{n-3}a_{n-2}a_{n-1}} - \frac{a_{n-1}x^2}{a_{n-4}a_{n-3}a_{n-2}} + \frac{a_{n-3}a_{n-1}}{a_{n-4}a_{n-2}}\right. \\
&\quad \left. - \frac{a_{n-3}x^2}{a_{n-4}a_{n-2}a_{n-1}} - \frac{a_{n-2}x^2}{a_{n-4}a_{n-3}a_{n-1}}\right) p_{n-1}(x).
\end{aligned} \tag{3.19}$$

It follows from Lemma 1, (3.17) and (3.19) that

$$\begin{aligned}
p_n'(x) &= (a_n a_{n+2}^2 a_{n+1}^2 + a_{n+1}^4 a_n + 2a_{n+1}^2 a_n^3 + a_{n+1}^2 a_{n-1}^2 a_n \\
&\quad + a_n^5 + 2a_n^3 a_{n-1}^2 + a_n a_{n-1}^4 + a_n a_{n-1}^2 a_{n-2}^2)\, p_{n-1}(x) \\
&\quad + a_n a_{n-1} a_{n-2}(a_{n+1}^2 + a_n^2 + a_{n-1}^2 + a_{n-2}^2 + a_{n-3}^2) \\
&\quad \times \left[\left(\frac{x^2}{a_{n-2}a_{n-1}} - \frac{a_{n-1}}{a_{n-2}}\right) p_{n-1}(x) - \frac{a_n x}{a_{n-2}a_{n-1}}\, p_n(x)\right] \\
&\quad + a_n a_{n-1} a_{n-2} a_{n-3} a_{n-4} \\
&\quad \times \left[\left(\frac{a_n a_{n-2} x}{a_{n-4}a_{n-3}a_{n-1}} + \frac{a_{n-3}a_n x}{a_{n-4}a_{n-2}a_{n-1}} - \frac{a_n x^3}{a_{n-4}a_{n-3}}\right) p_n(x)\right. \\
&\quad + \left(\frac{x^4}{a_{n-4}a_{n-3}a_{n-2}a_{n-1}} - \frac{a_{n-1}x^2}{a_{n-4}a_{n-3}a_{n-2}} + \frac{a_{n-3}a_{n-1}}{a_{n-4}a_{n-2}}\right. \\
&\quad \left.\left. - \frac{a_{n-3}x^2}{a_{n-4}a_{n-2}a_{n-1}} - \frac{a_{n-2}x^2}{a_{n-4}a_{n-3}a_{n-1}}\right) p_{n-1}(x)\right].
\end{aligned}$$

If we multiply this out and arrange the like terms, then we arrive at

$$\begin{aligned}
p_n'(x) &= a_n\,[a_{n+1}^2(a_{n+2}^2 + a_{n+1}^2 + a_n^2) + a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2) \\
&\quad + x^2(a_{n+1}^2 + a_n^2 + x^2)]\, p_{n-1}(x) \\
&\quad - a_n^2 x[a_n^2 + a_{n+1}^2 + a_{n-1}^2 + x^2] \cdot p_n(x).
\end{aligned} \tag{3.20}$$

By using (3.14) and (3.15), (3.20) can be rewritten as

$$p_n'(x) = a_n \phi_n(x)\, p_{n-1}(x) - \pi_n(x)\, p_n(x). \tag{3.21}$$

This proves the first half of Lemma 2.

By differentiating (3.21), we obtain

$$\begin{aligned} p_n''(x) = {} & [a_n \phi_n'(x)\, p_{n-1}(x) + a_n \phi_n(x)\, p_{n-1}'(x)] \\ & - [\pi_n'(x)\, p_n(x) + \pi_n(x)\, p_n'(x)]. \end{aligned} \tag{3.22}$$

Applying (3.21), (3.22) becomes

$$\begin{aligned} p_n''(x) = {} & a_n \phi_n'(x)\, p_{n-1}(x) \\ & + a_n \phi_n(x)[a_{n-1} \phi_{n-1}(x)\, p_{n-2}(x) - \pi_{n-1}(x)\, p_{n-1}(x)] \\ & - \pi_n'(x)\, p_n(x) - \pi_n(x)[a_n(x)\, \phi_n(x)\, p_{n-1}(x) - \pi_n(x)\, p_n(x)] \\ = {} & [a_n \phi_n'(x) - a_n \phi_n(x)\, \pi_{n-1}(x) - a_n \phi_n(x)\, \pi_n(x)]\, p_{n-1}(x) \\ & + [\pi_n^2(x) - \pi_n'(x)]\, p_n(x) + a_{n-1} a_n \phi_{n-1}(x)\, \phi_n(x)\, p_{n-2}(x). \end{aligned} \tag{3.23}$$

Now, applying (3.16) to the right-hand side of (3.23), we obtain

$$\begin{aligned} p_n''(x) = {} & [a_n \phi_n'(x) - a_n \phi_n(x)\, \pi_{n-1}(x) - a_n \phi_n(x)\, \pi_n(x)]\, p_{n-1}(x) \\ & + [\pi_n^2(x) - \pi_n'(x)]\, p_n(x) + a_{n-1} a_n \phi_{n-1}(x)\, \phi_n(x) \\ & \times \left[\frac{1}{a_{n-1}} (x p_{n-1}(x) - a_n\, p_n(x)) \right] \\ = {} & [a_n \phi_n'(x) - a_n \phi_n(x)\, \pi_{n-1}(x) - a_n \phi_n(x)\, \pi_n(x) \\ & + a_n \phi_{n-1}(x)\, \phi_n(x) x]\, p_{n-1}(x) \\ & + [\pi_n^2(x) - \pi_n'(x) - a_n^2 \phi_{n-1}(x)\, \phi_n(x)]\, p_n(x). \end{aligned} \tag{3.24}$$

Now, we are going to simplify the coefficinet of $p_{n-1}(x)$ in (3.24). From (3.14),

$$\phi_n'(x) = 2a_{n+1}^2 x + 2a_n^2 x + 4x^3.$$

Hence the coefficient of $p_{n-1}(x)$ in (3.24) can be written as

$$2a_{n+1}^2 a_n x + 2a_n^3 x + 4a_n x^3 - a_n \phi_n(x)[\pi_{n-1}(x) + \pi_n(x) - \phi_{n-1}(x) x].$$

But, with easy calculation,

$$\pi_{n-1}(x) + \pi_n(x) - \phi_{n-1}(x) x = -x^5,$$

hence, this coefficient equals

$$2a_n(a_{n+1}^2 + a_n^2)x + 4a_n x^3 + a_n x^5 \phi_n(x).$$

We have completed the proof of Lemma 2.

LEMMA 3. *There exists a constant $A > 0$ independent of n such that*

$$p_n(0) = A \cos\left(\frac{n\pi}{2}\right) n^{-1/12}\left[1 + O\left(\frac{1}{n}\right)\right], \tag{3.25}$$

and

$$p_n'(0) = 6 \cdot 10^{-5/6} \cdot A \cdot n^{3/4} \sin\left(\frac{n\pi}{2}\right)\left[1 + O\left(\frac{1}{n}\right)\right], \tag{3.26}$$

where $\{p_n(x)\}$ are the orthonormal polynomials associated with $\exp(-x^6/6)$.

Proof of Lemma 3. In order to prove (3.25), we consider two different cases:

Case 1. If n is odd, then $p_n(x)$ is an odd polynomial, since $\exp(-x^6/6)$ is even (see [13, p. 29]). Therefore $p_n(0) = 0$.

Case 2. If n is even, then by applying the recursion formula $xp_{n-1}(x) = a_n p_n(x) + a_{n-1} p_{n-2}(x)$ repeatedly with $x = 0$, we obtain

$$p_n(0) = -\frac{a_{n-1}}{a_n} p_{n-2}(0), \qquad p_{n-2}(0) = -\frac{a_{n-3}}{a_{n-2}},$$

$$p_{n-4}(0), \ldots, p_2(0) = -\frac{a_1}{a_2} p_0(0).$$

Therefore,

$$p_n(0) = (-1)^{n/2} \gamma_0 \prod_{k=1}^{n/2} a_{2k-1} \prod_{k=1}^{n/2} a_{2k}^{-1}, \tag{3.27}$$

or

$$p_n(0) = (-1)^{n/2}\gamma_0 \left[\prod_{k=1}^{n/2} a_{2k-1}\left(\frac{2k-1}{10}\right)^{-1/6} \Big/ \prod_{k=1}^{n/2} a_{2k}\left(\frac{2k}{10}\right)^{-1/6}\right]$$

$$\times (n!)^{1/6} \cdot 2^{-n/6} \cdot \left[\left(\frac{n}{2}\right)!\right]^{-1/3}. \tag{3.28}$$

The validity of (3.28) is based on the fact that

$$\begin{aligned}\prod_{k=1}^{n/2} (2k-1)^{-1/6} \prod_{k=1}^{n/2} (2k)^{1/6} &= \left(\frac{n!}{2\cdot 4\cdots n}\right)^{-1/6} \left[(2\cdot 1)\cdot(2\cdot 2)\cdots\left(2\cdot\frac{n}{2}\right)\right]^{1/6} \\ &= \left[\frac{n!}{2^{n/2}(n/2)!}\right]^{-1/6} (2^{n/2})^{1/6} \left[\left(\frac{n}{2}\right)!\right]^{1/6} \\ &= (n!)^{-1/6} 2^{n/6} \left[\left(\frac{n}{2}\right)!\right]^{1/3}.\end{aligned}$$

By (1.3), $a_k(k/10)^{-1/6} = (1+O(1/k^2))$ as $k\to\infty$, hence the two products in (3.28) converge as $n\to\infty$. Let us denote $\prod_{k=1}^{\infty} a_{2k-1}((2k-1)/10)^{-1/6}$ and $\prod_{k=1}^{\infty} a_{2k}^{-1}(2k/10)^{1/6}$ by α_1 and α_2, respectively, then we can rewrite (3.28) as

$$\begin{aligned}p_n(0) = (-1)^{n/2}\gamma_0 &\left[\frac{\alpha_1}{\prod_{k=n/2+1}^{\infty} a_{2k-1}((2k-1)/10)^{-1/6}}\right. \\ &\times \left.\frac{\alpha_2}{\prod_{k=n/2+1}^{\infty} a_{2k}(2k/10)^{-1/6}}\right]\cdot (n!)^{1/6}\cdot 2^{-n/6}\left[\left(\frac{n}{2}\right)!\right]^{-1/3}. \qquad (3.29)\end{aligned}$$

Since $a_{2k-1}((2k-1)/10)^{-1/6} = [1+O(1/(2k-1)^2)] = [1+O(1/k^2)]$, we have

$$\prod_{k=n/2+1}^{\infty} a_{2k-1}\left(\frac{2k-1}{10}\right)^{-1/6} = \prod_{k=n/2+1}^{\infty}\left[1+O\left(\frac{1}{k^2}\right)\right]. \qquad (3.30)$$

It follows that

$$\begin{aligned}\ln\left[\prod_{k=n/2+1}^{\infty} a_{2k-1}\left(\frac{2k-1}{10}\right)^{-1/6}\right] &= \sum_{k=n/2+1}^{\infty} \ln\left(1+O\left(\frac{1}{k^2}\right)\right) \\ &= \sum_{k=n/2+1}^{\infty} O\left(\frac{1}{k^2}\right). \qquad (3.31)\end{aligned}$$

But

$$\sum_{k=n/2+1}^{\infty} \frac{1}{k^2} \leqslant \int_{n/2}^{\infty} \frac{1}{t^2}\,dt = \frac{2}{n},$$

so from (3.31), we see that

$$\ln\left[\prod_{k=n/2+1}^{\infty} a_{2k-1}\left(\frac{2k-1}{10}\right)^{1/6}\right]=O\left(\frac{1}{n}\right).$$

Hence

$$\prod_{k=n/2+1}^{\infty} a_{2k-1}\left(\frac{2k-1}{10}\right)^{1/6}=e^{O(1/n)}$$

$$=1+O\left(\frac{1}{n}\right). \tag{3.32}$$

Similarly,

$$\prod_{k=n/2+1}^{\infty} a_{2k}\left(\frac{2k}{10}\right)^{-1/6}=1+O\left(\frac{1}{n}\right). \tag{3.33}$$

By combining (3.29), (3.32) and (3.33), we obtain

$$p_n(0)=(-1)^{n/2}\gamma_0\left[\frac{\alpha_1}{\alpha_2}\left(1+O\left(\frac{1}{n}\right)\right)\right](n!)^{1/6}\cdot 2^{-n/6}\cdot\left[\left(\frac{n}{2}\right)!\right]^{-1/3}. \tag{3.34}$$

By applying the Stirling formula, (3.34) becomes

$$\begin{aligned}p_n(0)&=(-1)^{n/2}\gamma_0\left[\frac{\alpha_1}{\alpha_2}\left(1+O\left(\frac{1}{n}\right)\right)\right]\left[e^{-n}n^n\sqrt{2\pi n}\left(1+O\left(\frac{1}{n}\right)\right)\right]^{1/6}\\&\quad\cdot 2^{-n/6}\cdot\left[e^{-n/2}\cdot\left(\frac{n}{2}\right)^{n/2}\sqrt{\pi n}\cdot\left(1+O\left(\frac{1}{n}\right)\right)\right]^{-1/3}\\&=(-1)^{n/2}\gamma_0\left(\frac{\alpha_1}{\alpha_2}\right)e^{-n/6}n^{n/6}(2\pi n)^{1/12}\left(1+O\left(\frac{1}{n}\right)\right)\\&\quad\cdot 2^{-n/6}\cdot e^{n/6}\left(\frac{n}{2}\right)^{-n/6}(\pi n)^{-1/6}\cdot\left(1+O\left(\frac{1}{n}\right)\right)\\&=(-1)^{n/2}\gamma_0\left(\frac{\alpha_1}{\alpha_2}\right)2^{1/12}(\pi n)^{-1/12}\left(1+O\left(\frac{1}{n}\right)\right).\end{aligned}$$

If we use "A" to denote $\gamma_0(\alpha_1/\alpha_2)\,2^{1/12}\pi^{-1/12}$, then from the result of case 1 and case 2, we see that

$$p_n(0)=A\cos\left(\frac{n\pi}{2}\right)n^{-1/12}\left(1+O\left(\frac{1}{n}\right)\right). \tag{3.35}$$

Now, we are going to show (3.26), the second half of Lemma 3.

From (3.12) or (3.20) with $x=0$, we have

$$p_n'(0)=a_n[a_{n+1}^2(a_{n+2}^2+a_{n+1}^2+a_n^2)+a_n^2(a_{n+1}^2+a_n^2+a_{n-1}^2)]\,p_{n-1}(0). \tag{3.36}$$

Using the facts that $a_n=(n/10)^{1/6}(1+O(1/n))$, $a_n^2=(n/10)^{1/3}(1+O(1/n))$ and applying (3.35), we can write (3.36) as

$$\begin{aligned} p_n'(0)&=\left(\frac{n}{10}\right)^{1/6}\left(1+O\left(\frac{1}{n}\right)\right)\left[6\left(\frac{n}{10}\right)^{2/3}\left(1+O\left(\frac{1}{n}\right)\right)\right]\\ &\quad\times\left[A\cos\left(\frac{(n-1)\pi}{2}\right)(n-1)^{-1/12}\cdot\left(1+O\left(\frac{1}{n}\right)\right)\right]\\ &=6\cdot 10^{-5/6}n^{3/4}A\sin\left(\frac{n\pi}{2}\right)\left(1+O\left(\frac{1}{n}\right)\right). \end{aligned}$$

The proof of Lemma 3 is complete.

The following lemma was proved by G. Freud [4]; we will need it to prove our theorems.

LEMMA 4. *Let* $\{p_n(x)\}$ *be the orthonormal polynomials associated with* $\exp(-x^6/6)$. *There exists a constant* B *independent of* x *and* n *such that*

$$\sum_{k=0}^{n-1} p_k^2(x)\leqslant Bn^{5/6}\exp(x^6/6),$$

for $n=1, 2,\ldots$ *and* $x\in R$.

LEMMA 5. *Let* $\{p_n(x)\}$ *be the orthonormal polynomials associated with* $\exp(-x^6/6)$, *and* $0<\varepsilon<1$ *be fixed. Then there exists a constant* c *depending on* ε *only such that*

$$|p_n(x)|\leqslant cn^{-1/12}\exp(x^6/12),$$

for $n=1, 2,\ldots$ *and* $|x|\leqslant\sqrt[6]{32n/5}\,\varepsilon$, x *real.*

Remark. A generalization of this lemma was proved by P. G. Nevai [10]. Also note that $\sqrt[6]{32n/5}$ is asymptotically equivalent to $2a_n$ or X_n, the greatest zero of $p_n(x)$ (see [3]).

Proof of Lemma 5. Applying Lemma 2, we obtain

$$\begin{aligned}
p_n'(x)\,p_{n-1}(x)-p_{n-1}'(x)\,p_n(x) &= [a_n\phi_n(x)\,p_{n-1}(x)-\pi_n(x)\,p_n(x)]\,p_{n-1}(x)\\
&\quad-[a_{n-1}\phi_{n-1}(x)\,p_{n-2}(x)-\pi_{n-1}(x)\,p_{n-1}(x)]\,p_n(x)\\
&= a_n\phi_n(x)\,p_{n-1}^2(x)-\pi_n(x)\,p_n(x)\,p_{n-1}(x)\\
&\quad-a_{n-1}\phi_{n-1}(x)\,p_{n-2}(x)\,p_n(x)+\pi_{n-1}(x)\,p_{n-1}(x)\,p_n(x),
\end{aligned}$$

which, by recursion formula $xp_{n-1}(x)=a_n\,p_n(x)+a_{n-1}\,p_{n-2}(x)$, can be written as

$$\begin{aligned}
p_n'(x)\,p_{n-1}(x)-p_{n-1}'(x)\,p_n(x) &= a_n\phi_n(x)\,p_{n-1}^2(x)-\pi_n(x)\,p_{n-1}(x)\,p_n(x)-p_n(x)\,\phi_{n-1}(x)\\
&\quad\cdot[xp_{n-1}(x)-a_n\,p_n(x)]+\pi_{n-1}(x)\,p_{n-1}(x)\,p_n(x)\\
&= a_n\phi_{n-1}(x)\,p_n^2(x)+a_n\phi_n(x)\,p_{n-1}^2(x)\\
&\quad+[\pi_{n-1}(x)-\pi_n(x)-x\phi_{n-1}(x)]\,p_n(x)\,p_{n-1}(x). \qquad (3.37)
\end{aligned}$$

Thus by the Christoffel–Darboux formula [13], we have

$$\begin{aligned}
\sum_{k=0}^{n-1} p_k^2(x) &= a_n[p_n'(x)\,p_{n-1}(x)-p_{n-1}'(x)\,p_n(x)]\\
&= a_n^2\phi_{n-1}(x)\,p_n^2(x)+a_n^2\phi_n(x)\,p_{n-1}^2(x)\\
&\quad+a_n[\pi_{n-1}(x)-\pi_n(x)-x\phi_{n-1}(x)]\,p_n(x)\,p_{n-1}(x). \qquad (3.38)
\end{aligned}$$

Using the definitions (3.14) and (3.15) of $\phi_n(x)$ and $\pi_n(x)$ to simplify $\pi_{n-1}(x)-\pi_n(x)-x\phi_{n-1}(x)$, we obtain, from (3.38),

$$\begin{aligned}
\sum_{k=0}^{n-1} p_k^2(x) &= a_n^2\phi_{n-1}(x)\,p_n^2(x)+a_n^2\phi_n(x)\,p_{n-1}^2(x)\\
&\quad+[-2a_n^3a_{n+1}^2x-2a_n^5x\\
&\quad-2a_n^3a_{n-1}^2x-2a_n^3x^3-a_nx^5]\,p_n(x)\,p_{n-1}(x). \qquad (3.39)
\end{aligned}$$

Dividing (3.39) by n and observing that $2p_n(x)\,p_{n-1}(x)\leqslant p_n^2(X)+p_{n-1}^2(x)$, we get

$$\begin{aligned}
&\frac{1}{n}\sum_{k=0}^{n-1} p_k^2(x)\\
&\quad=\frac{1}{n}\,a_n^2\phi_{n-1}(x)\,p_n^2(x)+\frac{1}{n}\,a_n^2\phi_n(x)\,p_{n-1}^2(x)+\frac{1}{n}\\
&\qquad\cdot[-2a_n^3a_{n+1}^2x-2a_n^5x-2a_n^3a_{n-1}^2x-2a_n^3x^3-a_nx^5]\,p_n(x)\,p_{n-1}(x)\\
&\quad\geqslant\frac{1}{n}\,a_n^2\phi_{n-1}(x)\,p_n^2(x)+\frac{1}{n}\,a_n^2\phi_n(x)\,p_{n-1}^2(x)+\frac{1}{n}\\
&\qquad\cdot\Big[-a_n^3a_{n+1}^2|x|-a_n^5|x|\\
&\qquad-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\frac{1}{2}a_n|x|^5\Big](p_n^2(x)+p_{n-1}^2(x))\\
&\quad\geqslant\frac{1}{n}\,[a_n^2\phi_{n-1}(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|\\
&\qquad-a_n^3|x|^3-\frac{1}{2}a_n|x|^5]\,p_n^2(x)+\frac{1}{n}\Big[a_n^2\phi_n(x)-a_n^3a_{n+1}^2|x|\\
&\qquad-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\frac{1}{2}a_n|x|^5\Big]\,p_{n-1}^2(x).
\end{aligned}\tag{3.40}$$

Since $a_n=(n/10)^{1/6}[1+O(1/n^2)]$, $a_{n+1}=(n/10)^{1/6}[1+1/6n+O(1/n^2)]$. We thus see that if n is sufficiently large, then

$$\frac{a_{n+1}}{a_n}=\frac{1+(1/6n)+O(1/n^2)}{1+O(1/n^2)}=\frac{n^2+(n/6)+O(1)}{n^2+O(1)}>1.\tag{3.41}$$

Hence $a_{n+1}>a_n$ if n is sufficiently large. Consequently $\phi_n(x)\geqslant\phi_{n-1}(x)$ by the definition of $\phi_n(x)$. Therefore we can replace $\phi_n(x)$ on the right of (3.40) by $\phi_{n-1}(x)$, and obtain, if n is sufficiently large,

$$\begin{aligned}
\frac{1}{n}\sum_{k=0}^{n-1} p_k^2(x)\geqslant{}&\frac{1}{n}\Big[a_n^2\phi_{n-1}(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3\\
&-\frac{1}{2}a_n|x|^5\Big]\,p_n^2(x)+\frac{1}{n}\Big[a_n^2\phi_{n-1}(x)-a_{n+1}^2a_n^3|x|-a_n^5|x|\\
&-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\frac{1}{2}a_n|x|^5\Big]\,p_{n-1}^2(x).
\end{aligned}\tag{3.42}$$

Now, we are going to simplify the expression inside the bracket on the right of (3.42). Since, by definition of $\phi_n(x)$,

$$\begin{aligned} a_n^2\phi_{n-1}(x) &= a_n^2[a_n^2(a_{n+1}^2+a_n^2+a_{n-1}^2)+a_{n-1}^2(a_n^2+a_{n-1}^2+a_{n-2}^2) \\ &\quad + x^2(a_n^2+a_{n-1}^2+x^2)] \\ &= a_n^4a_{n+1}^2+a_n^6+2a_n^4a_{n-1}^2+a_{n-1}^4a_n^2+a_n^2a_{n-1}^2a_{n-2}^2 \\ &\quad + a_n^4x^2+a_n^2a_{n-1}^2x^2+a_n^2x^4, \end{aligned}$$

we have

$$\begin{aligned} & a_n^2\phi_{n-1}(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\tfrac{1}{2}a_n|x|^5 \\ &= a_n^2x^4-\tfrac{1}{2}a_n|x|^5+a_n^4x^2+a_n^2a_{n-1}^2x^2-a_n^3|x|^3 \\ &\quad -a_n^3a_{n-1}^2|x|-a_n^5|x|-a_n^3a_{n+1}^2|x|+a_n^2a_{n-1}^2a_{n-2}^2+a_n^2a_{n-1}^4 \\ &\quad +2a_n^4a_{n-1}^2+a_n^6+a_n^4a_{n+1}^2 \\ &= a_n^6\left[\frac{x^4}{a_n^4}-\frac{|x|^5}{2a_n^5}+\frac{x^2}{a_n^2}+\frac{a_{n-1}^2x^2}{a_n^4}-\frac{|x|^3}{a_n^3}-\frac{a_{n-1}^2|x|}{a_n^3}-\frac{|x|}{a_n}\right. \\ &\quad \left.-\frac{a_{n+1}^2|x|}{a_n^3}+\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}+\frac{a_{n-1}^4}{a_n^4}+\frac{2a_{n-1}^2}{a_n^2}+1+\frac{a_{n+1}^2}{a_n^2}\right], \end{aligned} \tag{3.43}$$

which, by long division algorithm, can be written as

$$\begin{aligned} & a_n^2\phi_{n-1}(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\tfrac{1}{2}a_n|x|^5 \\ &= a_n^6\left\{\left[\frac{x^4}{a_n^4}+\frac{2x^2}{a_n^2}+\frac{2(a_n^2-a_{n-1}^2)}{a_n^3}|x|+\frac{6a_n^2+2a_{n+1}^2-2a_{n-1}^2}{a_n^2}\right]\right. \\ &\quad \cdot\left[1-\frac{|x|}{2a_n}\right]-\frac{6a_n^2+2a_{n+1}^2-2a_{n-1}^2}{a_n^2}+\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}+\frac{a_{n-1}^4}{a_n^4} \\ &\quad \left.+\frac{2a_{n-1}^2}{a_n^2}+1+\frac{a_{n+1}^2}{a_n^2}\right\} \\ &= a_n^6\left\{\left[\frac{x^4}{a_n^4}+\frac{2x^2}{a_n^2}+\frac{2(a_n^2-a_{n-1}^2)}{a_n^3}|x|+\frac{2(a_{n+1}^2-a_{n-1}^2)}{a_n^2}+6\right]\right. \\ &\quad \cdot\left[1-\frac{|x|}{2a_n}\right]-\left[6+\frac{2(a_{n+1}^2-a_{n-1}^2)}{a_n^2}\right]+\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}+\frac{a_{n-1}^4}{a_n^4} \\ &\quad \left.+\frac{2a_{n-1}^2}{a_n^2}+1+\frac{a_{n+1}^2}{a_n^2}\right\}. \end{aligned} \tag{3.44}$$

Since, with the aid of (3.41), the expression inside the first bracket of (3.44) is always positive if n is sufficiently large, we obtain

$$a_n^2\phi_{n-1}(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\tfrac{1}{2}a_n|x|^5$$
$$\geqslant a_n^6\left\{6\left(1-\frac{|x|}{2a_n}\right)-\left[6+\frac{2(a_{n+1}^2-a_{n-1}^2)}{a_n^2}\right]+\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}\right.$$
$$\left.+\frac{a_{n-1}^4}{a_n^4}+\frac{2a_{n-1}^2}{a_n^2}+1+\frac{a_{n+1}^2}{a_n^2}\right\}. \tag{3.45}$$

Inserting (3.45) into (3.42), we get

$$\frac{1}{n}\sum_{k=0}^{n-1}p_k^2(x)\geqslant\frac{a_n^6}{n}\left[6\left(1-\frac{|x|}{2a_n}\right)-\left(6+\frac{2(a_{n+1}^2-a_{n-1}^2)}{a_n^2}\right)\right.$$
$$\left.+\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}+\frac{a_{n-1}^4}{a_n^4}+\frac{2a_{n-1}^2}{a_n^2}+1+\frac{a_{n+1}^2}{a_n^2}\right]p_n^2(x)$$
$$+\frac{a_n^6}{n}\left[6\left(1-\frac{|x|}{2a_n}\right)-\left(6+\frac{2(a_{n+1}^2-a_{n-1}^2)}{a_n^2}\right)\right.$$
$$\left.+\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}+\frac{a_{n-1}^4}{a_n^4}+\frac{2a_{n-1}^2}{a_n^2}+1+\frac{a_{n+1}^2}{a_n^2}\right]p_{n-1}^2(x).$$

We now replace $|x|$ by its largest value $\sqrt[6]{32n/5\varepsilon}$ as stated in this lemma, we obtain

$$\frac{1}{n}\sum_{k=0}^{n-1}p_k^2(x)\geqslant\frac{a_n^6}{n}\left[6\left(1-\frac{\sqrt[6]{32n/5\varepsilon}}{2a_n}\right)-\left(6+\frac{2(a_{n+1}^2-a_{n-1}^2)}{a_n^2}\right)\right.$$
$$\left.+\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}+\frac{a_{n-1}^4}{a_n^4}+\frac{2a_{n-1}^2}{a_n^2}+1+\frac{a_{n+1}^2}{a_n^2}\right]p_n^2(x)$$
$$+\frac{a_n^6}{n}\left[\left(1-\frac{\sqrt[6]{32n/5\varepsilon}}{2a_n}\right)-\left(6+\frac{2(a_{n+1}^2-a_{n-1}^2)}{a_n^2}\right)\right.$$
$$\left.+\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}+\frac{a_{n-1}^4}{a_n^4}+\frac{2a_{n-1}^2}{a_n^2}+1+\frac{a_{n+1}^2}{a_n^2}\right]p_{n-1}^2(x). \tag{3.46}$$

Let $n\to\infty$, then we see that the coefficients of $p_n^2(x)$ and $p_{n-1}^2(x)$ both converge to $(6/10)(1-\varepsilon)$, where $1/10$ is the limit of a_n^6/n as $n\to\infty$. Hence for $n\geqslant n(\varepsilon)$, these coefficients will be greater than $\frac{1}{2}(1-\varepsilon)$. Therefore we have

$$\frac{1}{n}\sum_{k=0}^{n-1}p_k^2(x)\geqslant\frac{1}{2}(1-\varepsilon)[p_n^2(x)+p_{n-1}^2(X)],$$

if n is sufficiently large. This implies that

$$p_n^2(x)\leqslant p_n^2(x)+p_{n-1}^2(x)\leqslant\frac{2}{1-\varepsilon}\frac{1}{n}\sum_{k=0}^{n-1}p_k^2(x).$$

Applying Lemma 4, we obtain

$$p_n^2(x) \leqslant \frac{2}{1-\varepsilon} \cdot \frac{1}{n} \cdot c \cdot n^{5/6} \exp(x^6/6), \tag{3.47}$$

for some constant c. Lemma 5 now follows from (3.47).

LEMMA 6. *Let* $\{p_n(x)\}$ *be the orthonormal polynomials with respect to the weight function* $\exp(-x^6/6)$. *Let* $a_n = \gamma_{n-1}/\gamma_n$, *where* $\gamma_n > 0$ *denotes the leading coefficient of* $p_n(x)$, *and* $z(x) = p_n(x)\exp(-x^6/6)[\phi_n(x)]^{-1/2}$, *where* $\phi_n(x) = a_{n+1}^2(a_{n+2}^2 + a_{n+1}^2 + a_n^2) + a_n^2(a_{n+1}^2 + a_n^2 + a_{n-1}^2) + x^2(a_{n+1}^2 + a_n^2 + x^2)$. *Then* $z(x)$ *satisfies the differential equation*:

$$\begin{aligned} z''(x) + \Bigg[&-\frac{1}{4}x^{10} - \frac{1}{2}x^5\phi_n'(x)/\phi_n(x) + \frac{5}{2}x^4 - \frac{3}{4}\phi_n'(x)\,\phi_n'(x)/\phi_n^2(x) \\ &+ \frac{1}{2}\frac{\phi_n''(x)}{\phi_n(x)} + a_n^2\phi_n(x)\,\phi_{n-1}(x) + \pi_n'(x) - \pi_n^2(x) - \pi_n(x)\cdot x^5 \\ &- 4\pi_n(x)\cdot x^3/\phi_n(x) - \frac{2x\pi_n(x)\cdot a_n^2}{\phi_n(x)}\Bigg] z = 0, \end{aligned} \tag{3.48}$$

where $\pi_n(x) = a_n^2 x(a_n^2 + a_{n+1}^2 + a_{n-1}^2 + x^2)$.

Proof of Lemma 6. From Lemma 2 we know that

$$p_n'(x) = a_n\phi_n(x)\,p_{n-1}(x) - \pi_n(x)\,p_n(x) \tag{3.49}$$

and

$$\begin{aligned} p_n''(x) = {} & [2a_n(a_{n+1}^2 + a_n^2)\,x + 4a_nx^3 + a_nx^5\phi_n(x)]\,p_{n-1}(x) \\ & + [\pi_n^2(x) - \pi_n'(x) - a_n^2\phi_n(x)\,\phi_{n-1}(x)]\,p_n(x). \end{aligned} \tag{3.50}$$

To prove this theorem, at first, we need to find a differential equation satisfied by $p_n(x)$. This can be done by eliminating $p_{n-1}(x)$ from (3.49) and (3.50). By doing so, we obtain

$$\begin{aligned} & p_n'(x)\cdot[2(a_{n+1}^2 + a_n^2)x + 4x^3 + x^5\phi_n(x)] - p_n''(x)\cdot\phi_n(x) \\ &\quad = a_n\phi_n(x)[2(a_{n+1}^2 + a_n^2)x + 4x^3 + x^5\phi_n(x)]p_{n-1}(x) \\ &\qquad - \pi_n(x)[2(a_{n+1}^2 + a_n^2)x + 4x^3 + x^5\phi_n(x)]\,p_n(x) \\ &\qquad - \{[2a_n(a_{n+1}^2 + a_n^2)\,x\phi_n(x) + 4a_nx^3\phi_n(x) + a_nx^5\phi_n^2(x)] \\ &\qquad \cdot p_{n-1}(x) + [\pi_n^2(x)\,\phi_n(x) - \pi_n'(x)\,\phi_n(x) - a_n^2\phi_n(x)\,\phi_{n-1}(x)]\,p_n(x)\}, \end{aligned}$$

or

$$p_n'(x)[2(a_{n+1}^2+a_n^2)x+4x^3+x^5\phi_n(x)]-p_n''(x)\phi_n(x)$$
$$=[a_n^2\phi_n^2(x)\,\phi_{n-1}(x)+\pi_n'(x)\,\phi_n(x)-\pi_n^2(x)\,\phi_n(x)-\pi_n(x)\,\phi_n(x)x^5$$
$$-4\pi_n(x)x^3-2a_{n+1}^2x\pi_n(x)-2a_n^2x\pi_n(x)]\,p_n(x). \tag{3.51}$$

Dividing both sides of (3.51) by $\phi_n(x)$, noting that $\phi_n(x)$ is always positive since $a_n>0$, and $\phi_n(x)$ is an even polynomial, we get the desired polynomial as follows:

$$p_n''(x)-\left[2(a_{n+1}^2+a_n^2)\frac{x}{\phi_n(x)}+\frac{4x^3}{\phi_n(x)}+x^5\right]p_n'(x)$$
$$+\left[a_n^2\phi_n(x)\,\phi_{n-1}(x)+\pi_n'(x)-\pi_n^2(x)-\pi_n(x)x^5-4\frac{\pi_n(x)x^3}{\phi_n(x)}\right.$$
$$\left.-2\frac{a_{n+1}^2x\pi_n(x)}{\phi_n(x)}-2\frac{a_n^2x\pi_n(x)}{\phi_n(x)}\right]\cdot p_n(x)=0. \tag{3.52}$$

In order to prove this theorem, we must transform $p_n(x)$ in (3.52) to $z(x)$.
Since $z(x)=p_n(x)\exp(-x^6/12)\,\phi_n(x)^{-1/2}$,

$$p_n(x)=z(x)\exp(x^6/12)\,\phi_n(x)^{1/2}. \tag{3.53}$$

Differentiating (3.53), we have

$$p_n'(x)=\exp(x^6/12)\,\phi_n(x)^{1/2}z'(x)+\tfrac{1}{2}x^5\cdot\exp(x^6/12)\,\phi_n(x)^{1/2}z(x)$$
$$+\tfrac{1}{2}\exp(x^6/12)\,\phi_n(x)^{-1/2}\phi_n'(x)]\,z(x). \tag{3.54}$$

Differentiating (3.54), and combining the similar terms, we obtain

$$p_n''(x)=\exp(x^6/12)\,\phi_n^{1/2}(x)\,z''(x)+[\tfrac{1}{2}x^5\exp(x^6/12)\,\phi_n^{1/2}(x)$$
$$+\tfrac{1}{2}\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n'(x)+\tfrac{1}{2}x^5\exp(x^6/12)\,\phi_n^{1/2}(x)$$
$$+\tfrac{1}{2}\exp(x^6/12)\,\phi_n(x)^{-1/2}\phi_n'(x)]\,z'(x)+[\tfrac{5}{2}x^4\exp(x^6/12)\,\phi_n^{1/2}(x)$$
$$+\tfrac{1}{4}x^{10}\exp(x^6/12)\,\phi_n^{1/2}(x)$$
$$+\tfrac{1}{4}x^5\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n'(x)\,\tfrac{1}{4}x^5\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n'(x)$$
$$-\tfrac{1}{4}\exp(x^6/12)\,\phi_n^{-3/2}(x)\,\phi_n'(x)\,\phi_n'(x)$$
$$+\tfrac{1}{2}\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n''(x)]\,z(x)$$
$$=\exp(x^6/12)\,\phi_n^{1/2}(x)\,z''(x)+[x^5\exp(x^6/12)\,\phi_n^{1/2}(x)$$
$$+\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n'(x)]\,z'(x)+[\tfrac{5}{2}x^4\exp(x^6/12)\,\phi_n^{1/2}(x)$$

$$+\tfrac{1}{4}x^{10}\exp(x^6/12)\,\phi_n^{1/2}(x)+\tfrac{1}{2}x^5\exp(x^6/12)\,\phi_n^{-1/2}(x)\cdot\phi_n'(x)$$
$$-\tfrac{1}{4}\exp(x^6/12)\,\phi_n^{-3/2}(x)\,\phi_n'(x)\,\phi_n'(x)$$
$$+\tfrac{1}{2}\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n''(x)]\,z(x). \tag{3.55}$$

Plugging (3.53), (3.54), (3.55) into (3.52) and using the fact that

$$4x^3+2(a_{n+1}^2+a_n^2)=\phi_n'(x),$$

we see that the coefficient of $z'(x)$ becomes zero, and we have

$$\exp(x^6/12)\,\phi_n^{1/2}(x)\,z''(x)$$
$$+\left\{\frac{5}{2}x^4\exp(x^6/12)\,\phi_n^{1/2}(x)+\frac{1}{4}x^{10}\exp(x^6/12)\,\phi_n^{1/2}(x)\right.$$
$$+\frac{1}{2}x^5\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n'(x)$$
$$-\frac{1}{4}\exp(x^6/12)\,\phi_n^{-3/2}(x)\,\phi_n'(x)\,\phi_n'(x)$$
$$+\frac{1}{2}\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n''(x)$$
$$-\frac{1}{2}x^5\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n'(x)$$
$$-\frac{1}{2}\exp(x^6/12)\,\phi_n'(x)\,\phi_n'(x)\,\phi_n^{-3/2}(x)$$
$$-\frac{1}{2}x^{10}\exp(x^6/12)\,\phi_n^{1/2}(x)$$
$$-\frac{1}{2}x^5\exp(x^6/12)\,\phi_n^{-1/2}(x)\,\phi_n'(x)+\left[\left(a_n^2\phi_n(x)\,\phi_{n-1}(x)\right.\right.$$
$$+\pi_n'(x)-\pi_n^2(x)-\pi_n(x)x^5-\frac{4\pi_n(x)x^3}{\phi_n(x)}-\frac{2\pi_n(x)\,xa_{n+1}^2}{\phi_n(x)}$$
$$\left.\left.\left.-2\,\frac{\pi_n(x)xa_n^2}{\phi_n(x)}\right)\right]\exp(x^6/12)\,\phi_n^{1/2}(x)\right\}z(x)=0. \tag{3.56}$$

Combining the like terms and dividing (3.56) by $\exp(x^6/12)\phi_n^{1/2}(x)$, we arrive at

$$z''(x)+\left[-\frac{1}{4}x^{10}-\frac{1}{2}x^5\phi_n^{-1}(x)\,\phi_n'(x)+\frac{5}{2}x^4-\frac{3}{4}\phi_n'(x)\,\phi_n'(x)\,\phi_n^{-2}(x)\right.$$

$$+\frac{1}{2}\phi_n^{-1}(x)\,\phi_n''(x)+a_n^2\phi_n(x)\,\phi_{n-1}(x)+\pi_n'(x)-\pi_n^2(x)-\pi_n(x)x^5$$

$$\left.-\frac{4\pi_n(x)x^3}{\phi_n(x)}-\frac{2a_{n+1}^2x\pi_n(x)}{\phi_n(x)}-\frac{2a_n^2x\pi_n(x)}{\phi_n(x)}\right]z(x)=0, \tag{3.57}$$

which completes the proof of Lemma 6.

LEMMA 7. *There exists a constant* $c>0$ *such that for* $0<\varepsilon<\pi/2$

$$\limsup_{n\to\infty}\int_{\cos\varepsilon\leqslant|x|(32n/5)^{-1/6}\leqslant 1}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right]p_n^2(x)$$
$$\times\exp(-x^6/6)\,dx\leqslant c(1-\cos\varepsilon).$$

Proof of Lemma 7. From (3.40), we have

$$\frac{1}{n}\sum_{k=0}^{n-1}p_k^2(x)$$
$$\geqslant\frac{1}{n}\left(a_n^2\phi_{n-1}(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_a^3a_{n-1}^2|x|-a_n^3|x|^3\right.$$
$$\left.-\frac{1}{2}a_n|x|^5\right)p_n^2(x)+\frac{1}{n}\left(a_n^2\phi_n(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|\right.$$
$$\left.-a_n^3|x|^3-\frac{1}{2}a_n|x|^5\right)p_{n-1}^2(x). \tag{3.58}$$

Now we are going to show that if n is sufficiently large and $|x|\leqslant(32n/5)^{1/6}$ then the expression inside the second parenthesis on the right of (3.58) is positive. By the definition of $\phi_n(x)$, this expression can be written as

$$a_n^2\phi_n(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\tfrac{1}{2}a_n|x|^5$$
$$=a_n^2x^4+a_n^4x^2+a_n^2a_{n+1}^2x^2+a_n^4a_{n-1}^2+a_n^6+a_n^4a_{n+1}^2+a_n^4a_{n+1}^2+a_n^2a_{n+1}^4$$
$$+a_n^2a_{n+1}^2a_{n+2}^2-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\tfrac{1}{2}a_n|x|^5,$$

which, by the long division algorithm, can be transformed to

$$a_n^2\phi_n(x) - a_n^3 a_{n+1}^2 |x| - a_n^5 |x| - a_n^3 a_{n-1}^2 |x| - a_n^3 |x|^3 - \tfrac{1}{2} a_n |x|^5$$
$$= a_n^6 \left\{ \left(\frac{x^4}{a_n^4} + \frac{2x^2}{a_n^2} + \frac{2(a_n^2 - a_{n+1}^2)}{a_n^3} |x| + \frac{6a_n^2 - 2a_{n+1}^2 + 2a_{n-1}^2}{a_n^2} \right) \left(1 - \frac{|x|}{2a_n} \right) \right.$$
$$\left. - \frac{6a_n^2 - 2a_{n+1}^2 + 2a_{n-1}^2}{a_n^2} + \frac{a_{n-1}^2}{a_n^2} + 1 + \frac{2a_{n+1}^2}{a_n^2} + \frac{a_{n+1}^2 a_{n+2}^2}{a_n^4} + \frac{a_{n+1}^4}{a_n^4} \right\}. \tag{3.59}$$

To show that this is positive if n is sufficiently large, we first consider the expression inside the first parenthesis, which is

$$\frac{x^4}{a_n^4} + \frac{2x^2}{a_n^2} + \frac{2(a_n^2 - a_{n+1}^2)}{a_n^3} |x| + \frac{6a_n^2 - 2a_{n+1}^2 + 2a_{n-1}^2}{a_n^2}.$$

The first two terms of this expression are obviously positive. As for the last two terms, with aid of (3.41), and the assumption $|x| \leqslant (32n/5)^{1/6}$, we obtain

$$\frac{6a_n^2 - 2a_{n+1}^2 + 2a_{n-1}^2}{a_n^2} + \frac{2(a_n^2 - a_{n+1}^2)}{a_n^3} |x|$$
$$\geqslant 6 - \frac{2a_{n+1}^2}{a_n^2} + \frac{2a_{n-1}^2}{a_n^2} + \frac{2(a_n^2 - a_{n+1}^2)}{a_n^3} \cdot \left(\frac{32n}{5} \right)^{1/6}.$$

By the asymptotic of a_n,

$$\frac{6a_n^2 - 2a_{n+1}^2 + 2a_{n-1}^2}{a_n^2} + \frac{2(a_n^2 - a_{n+1}^2)}{a_n^3} |x|$$
$$\geqslant 6 - 2\left(1 + \frac{1}{3n} + O\left(\frac{1}{n^2} \right) \right)$$
$$+ 2\left(1 - \frac{1}{3n} + O\left(\frac{1}{n^2} \right) \right) + \left\{ 2\left[\left(\frac{n}{10} \right)^{1/6} \left(1 + O\left(\frac{1}{n^2} \right) \right) \right] \right.$$
$$\left. - 2\left[\left(\frac{n}{10} \right)^{-1/6} \left(1 + \frac{1}{3n} + O\left(\frac{1}{n^2} \right) \right) \right] \right\} \left(\frac{32n}{5} \right)^{1/6} = 6$$
$$- 2 - \frac{2}{3n} + O\left(\frac{1}{n^2} \right) + 2 - \frac{2}{3n} + O\left(\frac{1}{n^2} \right)$$
$$+ \left[-\frac{2}{3n} \left(\frac{n}{10} \right)^{-1/6} + O(n^{-13/6}) \right] \left(\frac{32n}{5} \right)^{1/6} = 6 - \frac{8}{3n} + O\left(\frac{1}{n^2} \right) > 0, \tag{3.60}$$

if n is sufficiently large.

Next, we consider the expression $1-|x|/2a_n$ in (3.59). Since $|x| \leqslant (32n/5)^{1/6}$, we have

$$1-\frac{|x|}{2a_n} \geqslant 1-\frac{1}{2a_n}\left(\frac{32n}{5}\right)^{1/6} = 1-\left(\frac{n}{10}\right)^{1/6}\frac{1}{a_n}$$
$$= 1-\frac{(n/10)^{1/6}}{(n/10)^{1/6}[1+(1/36)(1/n^2)+O(1/n^4)]}$$
$$= 1-\frac{n^2}{n^2+1/36+O(1/n^2)} > 0, \tag{3.61}$$

if n is sufficiently large. Finally, we consider the expression inside the last parenthesis of (3.58).

$$\frac{a_{n-1}^2}{a_n^2}+1+\frac{2a_{n+1}^2}{a_n^2}+\frac{a_{n+1}^2a_{n+2}^2}{a_n^4}+\frac{a_{n+1}^4}{a_n^4}-\frac{6a_n^2-2a_{n+1}^2+2a_{n-1}^2}{a_n^2}$$
$$=\left(1-\frac{1}{3n}+O\left(\frac{1}{n^2}\right)\right)+1+2\left(1+\frac{1}{3n}+O\left(\frac{1}{n^2}\right)\right)+1+\frac{1}{n}+1$$
$$+\frac{2}{3n}+O\left(\frac{1}{n^2}\right)-\left[6-2-\frac{2}{3n}+2-\frac{2}{3n}+O\left(\frac{1}{n^2}\right)\right]$$
$$=\frac{10}{3n}+O\left(\frac{1}{n^2}\right)>0, \text{ if } n \text{ is sufficiently large.} \tag{3.62}$$

As a consequence of (3.60), (3.61) and (3.62),

$$a_n^2\phi_n(x)-a_n^3a_{n-1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3-\tfrac{1}{2}a_n|x|^5>0,$$

if n is sufficiently large. It follows from (3.58) that

$$\frac{1}{n}\sum_{k=0}^{n-1} p_k^2(x) \geqslant \frac{1}{n}\,(a_n^2\phi_{n-1}(x)-a_n^3a_{n+1}^2|x|-a_n^5|x|-a_n^3a_{n-1}^2|x|-a_n^3|x|^3$$
$$-\frac{1}{2}a_n|x|^5)\,p_n^2(x), \text{ if } n \text{ is sufficiently large.}$$

Using (3.45), we see that

$$\frac{1}{n}\textstyle\sum_{k=0}^{n-1} p_k^2(x) \geqslant \frac{a_n^6}{n}\left[6\left(1-\frac{|x|}{2a_n}\right)\right]p_n^2(x)+\frac{a_n^6}{n}\left[\frac{a_{n-1}^2a_{n-2}^2}{a_n^4}+\frac{a_{n-1}^4}{a_n^4}+\frac{2a_{n-1}^2}{a_n^2}\right.$$
$$\left.+1+\frac{a_{n+1}^2}{a_n^2}-6-\frac{2(a_{n+1}^2-a_{n-1}^2)}{a_n^2}\right]p_n^2(x)$$

$$\geqslant \frac{6}{10}\left(1-\frac{|x|}{2a_n}\right) p_n^2(x) + \frac{a_n^6}{n}\left[1-\frac{1}{n}+O\left(\frac{1}{n^2}\right)+1-\frac{2}{3n}\right.$$
$$+O\left(\frac{1}{n^2}\right)+4\left(1-\frac{1}{3n}+O\left(\frac{1}{n^2}\right)\right)-5$$
$$\left.-\left(1+\frac{1}{3n}+O\left(\frac{1}{n^2}\right)\right)\right] p_n^2(x)$$
$$=\frac{6}{10}\left(1-\frac{|x|}{2a_n}\right) p_n^2(x)+\frac{a_n^6}{n}\left[-\frac{10}{3n}+O\left(\frac{1}{n^2}\right)\right] p_n^2(x)$$
$$\geqslant \frac{6}{10}\left(1-\frac{|x|}{2a_n}\right) p_n^2(x)+\left[\frac{1}{10}\left(\frac{-10}{n}\right)\right] p_n^2(x)$$
$$=\frac{6}{10}\left(1-\frac{|x|}{2a_n}\right) p_n^2(x)-\frac{1}{n} p_n^2(x). \tag{3.63}$$

While proving the preceding inequality, we have used the fact that $a_n^6/n \geqslant 1/10$ if n is sufficiently large, which can be easily seen from the asymptotic of a_n. From (3.63), we have

$$\frac{1}{n}\sum_{k=0}^{n} p_k^2(x) \geqslant \frac{3}{5}\left(1-\frac{|x|}{2a_n}\right) p_n^2(x)$$
$$=\frac{3}{5}\frac{(1-|x|/2a_n)(1+|x|/2a_n)}{1+|x|/2a_n} p_n^2(x)$$
$$=\frac{3}{5}\frac{1-x^2/4a_n^2}{1+|x|/2a_n} p_n^2(x). \tag{3.64}$$

From the asymptotic of a_n, it is easy to see that $4a_n^2 \geqslant (32n/5)^{1/3}$ if n is sufficiently large, and from (3.61), $|x|/2a_n \leqslant 1$. Therefore (3.64) implies

$$\frac{1}{n}\sum_{k=0}^{n} p_k^2 \geqslant \frac{3}{5}\cdot\frac{1}{2}\left[1-\left(\frac{32n}{5}\right)^{1/3} x^2\right] p_n^2(x),$$

or

$$\left[1-\left(\frac{32n}{5}\right)^{-1/3} x^2\right] p_n^2(x) \leqslant \frac{10}{3n}\sum_{k=0}^{n} p_k^2(x).$$

Applying Lemma 4 to the right-hand side of preceding inequality, we get

$$\left[1-\left(\frac{32n}{5}\right)^{-1/3} x^2\right] p_n^2(x) \leqslant \frac{10}{3n}\sum_{k=0}^{n} p_k^2(x)$$
$$\leqslant Bn^{-1/6}\exp(x^6/6),$$

where B is a constant.

Therefore,

$$\left[1 - x^2 \left(\frac{32n}{5}\right)^{-1/3}\right] p_n^2(x) \exp(-x^6/6) \leqslant Bn^{-1/6},$$

and

$$\limsup_{n \to \infty} \int_{\cos \varepsilon \leqslant |x|(32n/5)^{-1/6} \leqslant 1} \left[1 - x^2 \left(\frac{32n}{5}\right)^{-1/3}\right] p_n^2(x) \exp(-x^6/6)\, dx$$
$$\leqslant \limsup_{n \to \infty} n^{-1/6} \left[\left(\frac{32n}{5}\right)^{1/6} - (\cos \varepsilon) \left(\frac{32n}{5}\right)^{1/6}\right] \cdot B$$
$$\leqslant c(1 - \cos \varepsilon),$$

where c is a constant. Thus the proof of Lemma 7 is complete.

From Lemma 6, we know that $z(x) = p_n(x) \exp(-x^6/12)\, \phi_n^{-1/2}(x)$ satisfies the differential equation:

$$z'' + \left[-\frac{1}{4} x^{10} - \frac{1}{2} x^5 \phi_n^{-1}(x)\, \phi_n'(x) + \frac{5}{2} x^4 - \frac{3}{4} \phi_n'(x)\, \phi_n'(x)\, \phi_n^{-2}(x)\right.$$
$$+ \frac{1}{2} \phi_n^{-1}(x)\, \phi_n''(x) + a_n^2 \phi_n(x)\, \phi_{n-1}(x) + \pi_n'(x) - \pi_n^2(x) - \pi_n(x) x^5$$
$$\left. - 4 \frac{\pi_n(x) \cdot x^3}{\phi_n(x)} - 2 \cdot \frac{x \cdot \pi_n(x) \cdot a_{n+1}^2}{\phi_n(x)} - 2 \cdot \frac{a_n^2 \cdot x \cdot \pi_n(x)}{\phi_n(x)}\right] z = 0, \quad (3.65)$$

where

$$\phi_n(x) = a_{n+1}^2 (a_n^2 + a_{n+1}^2 + a_{n+2}^2) + a_n^2 (a_{n-1}^2 + a_n^2 + a_{n+1}^2) + x^2 (a_{n+1}^2 + a_n^2 + x^2),$$

and

$$\pi_n(x) = a_n^2 x (a_{n-1}^2 + a_n^2 + a_{n+1}^2 + x^2).$$

Thus, if we define s_n as

$$s_n = a_n^2 (a_{n-1}^2 + a_n^2 + a_{n+1}^2), \quad (3.66)$$

then we can express $\phi_n(x)$ and $\pi_n(x)$ as

$$\phi_n(x) = s_{n+1} + s_n + x^2 (a_{n+1}^2 + a_n^2 + x^2), \quad (3.67)$$
$$\pi_n(x) = s_n x + a_n^2 x^3.$$

In the following lemma, we will investigate the asymptotic behavior of the coefficient of z in (3.65), which will be denoted as $f_n(x)$.

LEMMA 8. *Let $0 < \varepsilon < \pi/2$ be fixed and let g be defined by*

$$g(\theta) = -\frac{1}{10}\cos 6\theta - \frac{2}{5}\cos 4\theta - \frac{1}{2}\cos 2\theta + 1, \tag{3.68}$$

then we have

$$4\left(\frac{n}{10}\right)^{1/3} \sin^2\theta \cdot f_n(x) = \left[ng(\theta) + \frac{1}{2}\right]^2 + O(1) \tag{3.69}$$

uniformly for $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$, where $x = 2(n/10)^{1/6}\cos\theta$.

Proof of Lemma 8. First, we will find the asymptotics for some basic expressions which we need when we proceed with the proof. Basically, all these asymptotics are derived from the asymptotic of a_n and the above given condition $x = 2(n/10)^{1/6}\cos\theta$.

$$\begin{aligned}
s_n &= a_n^2(a_{n-1}^2 + a_n^2 + a_{n+1}^2) \\
&= \left(\frac{n}{10}\right)^{1/3}\left(1 + O\left(\frac{1}{n^2}\right)\right)\left[\left(\frac{n}{10}\right)^{1/3}\left(1 - \frac{1}{3n} + O\left(\frac{1}{n^2}\right)\right)\right. \\
&\quad + \left(\frac{n}{10}\right)^{1/3}\left(1 + O\left(\frac{1}{n^2}\right)\right) \\
&\quad \left. + \left(\frac{n}{10}\right)^{1/3}\left(1 + \frac{1}{3n} + O\left(\frac{1}{n^2}\right)\right)\right] \\
&= \left(\frac{n}{10}\right)^{2/3}\left(3 + O\left(\frac{1}{n^2}\right)\right),
\end{aligned} \tag{3.70}$$

$$\begin{aligned}
\pi_n(x) &= s_n x + a_n^2 x^3 \\
&= \left(\frac{n}{10}\right)^{2/3}\left(3 + O\left(\frac{1}{n^2}\right)\right) \cdot 2\left(\frac{n}{10}\right)^{1/6}\cos\theta \\
&\quad + \left[\left(\frac{n}{10}\right)^{1/3}\left(1 + O\left(\frac{1}{n^2}\right)\right)\right] \cdot 2^3 \cdot \left(\frac{n}{10}\right)^{1/2}\cos^3\theta \\
&= 2\left(\frac{n}{10}\right)^{5/6}\cos\theta\left(3 + O\left(\frac{1}{n^2}\right)\right) \\
&\quad + 8\left(\frac{n}{10}\right)^{5/6}\cos^3\theta\left(1 + O\left(\frac{1}{n^2}\right)\right) \\
&= \left(\frac{n}{10}\right)^{5/6}(6\cos\theta + 8\cos^3\theta) + O(n^{-7/6}),
\end{aligned} \tag{3.71}$$

$$
\begin{aligned}
\phi_n(x) &= s_{n+1} + s_n + x^2(a_{n+1}^2 + a_n^2 + x^2) \\
&= \left(\frac{n}{10}\right)^{2/3}\left(3 + \frac{2}{n} + O\left(\frac{1}{n^2}\right)\right) + \left(\frac{n}{10}\right)^{2/3}\left(3 + O\left(\frac{1}{n^2}\right)\right) \\
&\quad + 4\left(\frac{n}{10}\right)^{1/3}\cos^2\theta \\
&\quad \cdot\left[\left(\frac{n}{10}\right)^{1/3}\left(1 + \frac{1}{3n} + O\left(\frac{1}{n^2}\right)\right) + \left(\frac{n}{10}\right)^{1/3}\left(1 + O\left(\frac{1}{n^2}\right)\right)\right. \\
&\quad \left. + 4\left(\frac{n}{10}\right)^{1/3}\cos^2\theta\right] \\
&= 6\left(\frac{n}{10}\right)^{2/3} + 2\left(\frac{n}{10}\right)^{2/3}\frac{1}{n} + O(n^{-4/3}) \\
&\quad + 8\left(\frac{n}{10}\right)^{2/3}\cos^2\theta + \frac{4}{3}\left(\frac{n}{10}\right)^{2/3} \\
&\quad \cdot\frac{1}{n}\cos^2\theta + 4\left(\frac{n}{10}\right)^{1/3}\cos^2\theta\cdot O(n^{-5/3}) + 16\left(\frac{n}{10}\right)^{2/3}\cos^4\theta \\
&= \left(\frac{n}{10}\right)^{2/3}[8\cos^2\theta + 16\cos^4\theta + 6] + O(n^{-1/3}),
\end{aligned} \tag{3.72}
$$

$$
\begin{aligned}
\phi_n'(x) &= 2x(a_{n+1}^2 + a_n^2) + 4x^3 \\
&= 2\cdot 2\left(\frac{n}{10}\right)^{1/6}\cos\theta\left[\left(\frac{n}{10}\right)^{1/3}\left(1 + \frac{1}{3n} + O\left(\frac{1}{n^2}\right)\right)\right. \\
&\quad \left. + \left(\frac{n}{10}\right)^{1/3}\left(+ O\left(\frac{1}{n^2}\right)\right)\right] + 4\cdot 2^3\left(\frac{n}{10}\right)^{1/2}\cos^3\theta \\
&= 4\left(\frac{n}{10}\right)^{1/6}\cos\theta\left[2\left(\frac{n}{10}\right)^{1/3} + \frac{1}{3n}\left(\frac{n}{10}\right)^{1/3}\right. \\
&\quad \left. + O(n^{-5/3})\right] + 32\left(\frac{n}{10}\right)^{1/2}\cos^3\theta \\
&= 8\left(\frac{n}{10}\right)^{1/2}\cos\theta + \frac{4}{3n}\left(\frac{n}{10}\right)^{1/2}\cos\theta \\
&\quad + O(n^{-3/2}) + 32\left(\frac{n}{10}\right)^{1/2}\cos^3\theta \\
&= \left(\frac{n}{10}\right)^{1/2}[8\cos\theta + 32\cos^3\theta] + O(n^{-1/2})
\end{aligned} \tag{3.73}
$$

and

$$\begin{aligned}\phi_n''(x) &= 2(a_{n+1}^2 + a_n^2) + 12x^2 \\ &= 2\left[2\left(\frac{n}{10}\right)^{1/3} + \frac{1}{3n}\left(\frac{n}{10}\right)^{1/3} + O(n^{-5/3})\right] + 12\cdot 2^2\left(\frac{n}{10}\right)^{1/3}\cos^2\theta \\ &= \left(\frac{n}{10}\right)^{1/3}[4 + 48\cos^2\theta] + O(n^{-2/3}). \end{aligned} \tag{3.74}$$

We will now find the asymptotic for each term in $f_n(x)$, using the above basic asymptotics.

1. The first term:

$$\begin{aligned}-\frac{1}{4}x^{10} &= -\frac{1}{4}\left(2\left(\frac{n}{10}\right)^{1/6}\cos\theta\right)^{10} \\ &= -2^8\left(\frac{n}{10}\right)^{5/3}\cos^{10}\theta.\end{aligned}$$

2. The second term:

$$\begin{aligned}&-\frac{1}{2}x^5\phi_n^{-1}(x)\,\phi_n'(x) \\ &= \frac{-\frac{1}{2}x^5[4x^3 + 2(a_{n+1}^2 + a_n^2)x]}{s_{n+1} + s_n + x^4 + (a_{n+1}^2 + a_n^2)x^2} \\ &= -\frac{1}{2}\cdot\frac{2^5(n/10)^{5/6}\cos^5\theta[(n/10)^{1/2}(8\cos\theta + 32\cos^3\theta) + O(n^{-1/2})]}{(n/10)^{2/3}[8\cos^2\theta + 16\cos^4\theta + 6] + O(n^{-1/3})} \\ &= -\frac{1}{2}\left[2^5\left(\frac{n}{10}\right)^{4/3}\cos^5\theta(8\cos\theta + 32\cos^3\theta) + O(n^{1/3})\right] \\ &\quad\cdot\left[\left(\frac{n}{10}\right)^{-2/3}(8\cos^2\theta + 16\cos^4\theta + 6)^{-1} + O(n^{-5/3})\right] \\ &= -2^4\left(\frac{n}{10}\right)^{2/3}\frac{\cos^5\theta(8\cos\theta + 32\cos^3\theta)}{8\cos^2\theta + 16\cos^4\theta + 6} + O(n^{-1/3}).\end{aligned}$$

3. The third term:

$$\begin{aligned}\frac{5}{2}x^4 &= \frac{5}{2}\cdot 2^4\left(\frac{n}{10}\right)^{2/3}\cos^4\theta \\ &= 40\cdot\left(\frac{n}{10}\right)^{2/3}\cos^4\theta.\end{aligned}$$

4. The fourth term:

$$-\tfrac{3}{4}\phi_n'(x)\,\phi_n'(x)\,\phi_n^{-2}(x)=O(n^{-1/3}).$$

This follows immediately from (3.72) and (3.73).

5. The fifth term:

$$\tfrac{1}{2}\phi_n^{-1}(x)\,\phi_n''(x)=O(n^{-1/3}).$$

This follows from (3.72) and (3.74).

6. The next four terms:

$$\begin{aligned}
&a_n^2\phi_n(x)\,\phi_{n-1}(x)+\pi_n'(x)-\pi_n^2(x)-\pi_n(x)\cdot x^5\\
&\quad=n\left[1+O\left(\frac{1}{n^2}\right)\right]\cdot 2^4\left(\frac{n}{10}\right)^{2/3}\cos^4\theta\\
&\qquad+\left[15\left(\frac{n}{10}\right)^{4/3}\left(1+O\left(\frac{1}{n^2}\right)\right)+3\left(\frac{n}{10}\right)^{1/3}\left(1+O\left(\frac{1}{n^2}\right)\right)\right]\\
&\qquad\cdot 2^2\left(\frac{n}{10}\right)^{1/3}\cos^2\theta+\left[36\left(\frac{n}{10}\right)^{5/3}\left(1+O\left(\frac{1}{n^2}\right)\right)\right.\\
&\qquad\left.+3\left(\frac{n}{10}\right)^{2/3}\left(1+O\left(\frac{1}{n^2}\right)\right)\right]\\
&\quad=2^4\cdot n\left(\frac{n}{10}\right)^{2/3}\cos^4\theta+O(n^{-1/3})\\
&\qquad+60\left(\frac{n}{10}\right)^{5/3}\cos^2\theta+O(n^{-1/3})+12\left(\frac{n}{10}\right)^{2/3}\cos^2\theta\\
&\qquad+O(n^{-4/3})+36\left(\frac{n}{10}\right)^{5/3}+O(n^{-1/3})+3\left(\frac{n}{10}\right)^{2/3}+O(n^{-4/3})\\
&\quad=\left(\frac{n}{10}\right)^{5/3}[160\cos^4\theta+60\cos^2\theta+36]\\
&\qquad+\left(\frac{n}{10}\right)^{2/3}[12\cos^2\theta+3]+O(n^{-1/3}).
\end{aligned}$$

7. The next three terms, which can be written as

$$\frac{-4\pi_n(x)x^3-2x\pi_n(x)a_{n+1}^2-2xa_n^2\pi_n(x)}{\phi_n(x)}.$$

Let us find the asymptotic for numerator first.

$$
\begin{aligned}
&-4\pi_n(x)x^3-2x\pi_n(x)a_{n+1}^2-2x\pi_n(x)a_n^2\\
&\quad=-4\left[\left(\frac{n}{10}\right)^{5/6}(6\cos\theta+8\cos^3\theta)+O(n^{-7/6})\right]\left\{2^3\left(\frac{n}{10}\right)^{1/2}\cos^3\theta\right.\\
&\qquad-4\left(\frac{n}{10}\right)^{1/6}\cos\theta\left[\left(\frac{n}{10}\right)^{1/3}\cdot\left(1+\frac{1}{3n}+O\left(\frac{1}{n^2}\right)\right)\right]\\
&\qquad\left.-4\left(\frac{n}{10}\right)^{1/6}\cos\theta\left[\left(\frac{n}{10}\right)^{1/3}\left(1+O\left(\frac{1}{n^2}\right)\right)\right]\right\}\\
&\quad=\left[\left(\frac{n}{10}\right)^{5/6}(6\cos\theta+8\cos^3\theta)+O(n^{-7/6})\right]\\
&\qquad\cdot\left[-32\left(\frac{n}{10}\right)^{1/2}\cos^3\theta-8\left(\frac{n}{10}\right)^{1/2}\cdot\cos\theta+O(n^{-1/2})\right]\\
&\quad=-32\left(\frac{n}{10}\right)^{8/6}(6\cos\theta+8\cos^3\theta)\cos^3\theta+O(n^{-2/3})-8\left(\frac{n}{10}\right)^{8/6}\\
&\qquad\cdot(6\cos\theta+8\cos^3\theta)\cos\theta+O(n^{-2/3})+O(n^{1/3})\\
&\quad=\left(\frac{n}{10}\right)^{4/3}(-192\cos^4\theta-256\cos^6\theta-48\cos^2\theta-64\cos^4\theta)+O(n^{1/3})\\
&\quad=\left(\frac{n}{10}\right)^{4/3}(-256\cos^6\theta-256\cos^4\theta-48\cos^2\theta)+O(n^{1/3}).
\end{aligned}
$$

So from (3.72), we see that

$$
\begin{aligned}
&\frac{-4\pi_n(x)x^3-2x\pi_n(x)a_{n+1}^2-2x\pi_n(x)a_n^2}{\phi_n(x)}\\
&\quad=\left[\left(\frac{n}{10}\right)^{4/3}(-256\cos^6\theta-256\cos^4\theta-48\cos^2\theta)+O\left(\frac{1}{n^3}\right)\right]\\
&\qquad\cdot\left[\left(\frac{n}{10}\right)^{-2/3}(8\cos^2\theta+16\cos^4\theta+6)^{-1}+O(n^{-5/3})\right]\\
&\quad=\left(\frac{n}{10}\right)^{2/3}\frac{-256\cos^6\theta-256\cos^4\theta-48\cos^2\theta}{16\cos^4\theta+8\cos^2\theta+6}+O(n^{-1/3}).
\end{aligned}
$$

To get the asymptotic for $f_n(x)$, we must add all the above asymptotics from case 1 to case 7 together. To simplify the computation, let us first add those which contain the factor $(n/10)^{2/3}$. Thus, we obtain, from cases 2, 3, 6 and 7,

$$
\begin{aligned}
&-2^4\left(\frac{n}{10}\right)^{2/3}\frac{\cos^5\theta(8\cos\theta+32\cos^3\theta)}{8\cos^2\theta+16\cos^4\theta+6}\\
&\quad+O(n^{-1/3})+40\left(\frac{n}{10}\right)^{2/3}\cos^4\theta\\
&\quad+\left(\frac{n}{10}\right)^{2/3}(12\cos^2\theta+3)+O(n^{-1/3})\\
&\quad+\left(\frac{n}{10}\right)^{2/3}\frac{-256\cos^6\theta-256\cos^4 4\,\theta-48\cos^2\theta}{16\cos^4\theta+8\cos^2\theta+6}+O(n^{-1/3})\\
&=\left(\frac{n}{10}\right)^{2/3}[-128\cos^6\theta-512\cos^8\theta+40\cos^4\theta\\
&\quad\cdot(8\cos^2\theta+16\cos^4\theta+6)+(12\cos^2\theta+3)\\
&\quad\cdot(8\cos^2\theta+16\cos^4\theta+6)-256\cos^6\theta-256\cos^4\theta\\
&\quad-48\cos^2\theta]/(16\cos^4\theta+8\cos^2\theta+6)+O(n^{-1/3})\\
&=\left(\frac{n}{10}\right)^{2/3}\frac{128\cos^8\theta+128\cos^6\theta+128\cos^4\theta+48\cos^2\theta+18}{16\cos^4\theta+8\cos^2\theta+6}\\
&\quad+O(n^{-1/3})\\
&=\left(\frac{n}{10}\right)^{2/3}\frac{\frac{1}{2}(16\cos^4\theta+8\cos^2\theta+6)^2}{16\cos^4\theta+8\cos^2\theta+6}+O(n^{-1/3})\\
&=\frac{1}{2}\left(\frac{n}{10}\right)^{2/3}(16\cos^4+8\cos^2\theta+6)+O(n^{-1/3}).
\end{aligned}
\tag{3.75}
$$

Next, we add those asymptotics which contain the factor $(n/10)^{5/3}$. We thus obtain, from the above cases 1 and 6,

$$
\begin{aligned}
&-2^8\left(\frac{n}{10}\right)^{5/3}\cos^{10}\theta+\left(\frac{n}{10}\right)^{5/3}(160\cos^4\theta+60\cos^2\theta+36)+O(n^{-1/3})\\
&=\left(\frac{n}{10}\right)^{5/3}(160\cos^4\theta+60\cos^2\theta+36-256\cos^{10}\theta)+O(n^{-1/3})\\
&=\left(\frac{n}{10}\right)^{5/3}(1-\cos^2\theta)(256\cos^8\theta+256\cos^6\theta\\
&\quad+256\cos^4\theta+96\cos^2\theta+36)+O(n^{-1/3})\\
&=\left(\frac{n}{10}\right)^{5/3}\sin^2\theta(16\cos^4\theta+8\cos^2\theta+6)^2+O(n^{-1/3}).
\end{aligned}
\tag{3.76}
$$

Combining (3.75) and (3.76), we get

$$f_n(x) = \frac{1}{2}\left(\frac{n}{10}\right)^{2/3} (16 \cos^4 \theta + 8 \cos^2 \theta + 6) + \left(\frac{n}{10}\right)^{5/3} \sin^2 \theta \cdot (16 \cos^4 \theta + 8 \cos^2 \theta + 6)^2 + O(n^{-1/3}). \tag{3.77}$$

It follows that

$$\begin{aligned} 4\left(\frac{n}{10}\right)^{1/3} &\sin^2 \theta \cdot f_n(x) \\ &= 2\left(\frac{n}{10}\right) \sin^2 \theta (16 \cos^4 \theta + 8 \cos^2 \theta + 6) + 4\left(\frac{n}{10}\right)^2 \sin^4 \theta \\ &\quad \cdot (16 \cos^4 \theta + 8 \cos^2 \theta + 6)^2 + O(1) \\ &= \left[2\left(\frac{n}{10}\right) \sin^2 \theta (16 \cos^4 \theta + 8 \cos^2 \theta + 6) + \frac{1}{2}\right]^2 + O(1) \\ &= \left[\frac{n}{5} \sin^2 \theta (16 \cos^4 \theta + 8 \cos^2 \theta + 6) + \frac{1}{2}\right]^2 + O(1). \end{aligned} \tag{3.78}$$

Comparing (3.69) with (3.78). We see that if we can show

$$\begin{aligned} \tfrac{1}{5} &\sin^2 \theta (16 \cos^4 \theta + 8 \cos^2 \theta + 6) \\ &= -\tfrac{1}{10} \cos 6\theta - \tfrac{2}{5} \cos 4\theta - \tfrac{1}{2} \cos 2\theta + 1, \end{aligned} \tag{3.79}$$

then the proof of this lemma will be established. Indeed,

$$\begin{aligned} \frac{1}{5} &\sin^2 \theta (16 \cos^4 \theta + 8 \cos^2 \theta + 6) \\ &= \frac{1}{5} \frac{1 - \cos 2\theta}{2} \left[16\left(\frac{1 + \cos 2\theta}{2}\right)^2 + 8\left(\frac{1 + \cos 2\theta}{2}\right) + 6\right] \\ &= \frac{1}{5} \frac{1 - \cos 2\theta}{2} \left[16\left(\frac{1}{8} \cos 4\theta + \frac{1}{2} \cos 2\theta + \frac{3}{8}\right)\right. \\ &\quad \left. + 8\left(\frac{1 + \cos 2\theta}{2}\right) + 6\right] \\ &= \frac{1}{5} (1 - \cos 2\theta)(\cos 4\theta + 4 \cos 2\theta + 8 + 2 \cos 2\theta) \end{aligned}$$

$$=\frac{1}{5}(\cos 4\theta+4\cos 2\theta+8+2\cos 2\theta$$

$$-\cos 2\theta\cos 4\theta-6\cos^2 2\theta-8\cos 2\theta)$$

$$=\frac{1}{5}\left[\cos 4\theta+6\cos 2\theta+8-\frac{1}{2}(\cos 6\theta+\cos 2\theta)\right.$$

$$\left.-6\cdot\frac{1+\cos 4\theta}{2}-8\cos 2\theta\right]$$

$$=-\frac{1}{10}\cos 6\theta-\frac{2}{5}\cos 4\theta-\frac{1}{2}\cos 2\theta+1.$$

Thus, the proof is complete.

Now we are in a position to prove Theorem 1.

4. Proofs

Proof of Theorem 1. We decompose the proof of this theorem into four stages.

Stage 1. Let us recall that the function $z(x)=p_n(x)\exp(-x^6/12)\cdot[\phi_n(x)]^{-1/2}$ satisfies a differential equation

$$z''+f_n(x)z=0, \tag{4.1}$$

where

$$f_n(x)=-\frac{1}{4}x^{10}-\frac{1}{2}x^5\phi_n^{-1}(x)+\frac{5}{2}x^4-\frac{3}{4}\phi_n'(x)\,\phi_n'(x)\,\phi_n^{-2}(x)$$
$$+\frac{1}{2}\phi_n^{-1}(x)\,\phi_n''(x)+a_n^2\phi_n(x)\,\phi_{n-1}(x)+\pi_n'(x)-\pi_n^2(x)-\pi_n(x)x^5$$
$$-\frac{4\pi_n(x)\cdot x^3}{\phi_n(x)}-\frac{2x\pi_n(x)\cdot a_{n+1}^2}{\phi_n(x)}-\frac{2x\pi_n(x)\cdot a_n^2}{\phi_n(x)}. \tag{4.2}$$

Now let

$$x=\left(\frac{32n}{5}\right)^{1/6}\cos\theta=2\left(\frac{n}{10}\right)^{1/6}\cos\theta, \tag{4.3}$$

and

$$u(\theta)=z(x)=z\left(2\left(\frac{n}{10}\right)^{1/6}\cos\theta\right). \tag{4.4}$$

In this stage, we will transform the differential equation (4.1) to a differential equation in terms of u and θ. From (4.3) and (4.4), we see that

$$\frac{du}{d\theta}=\frac{dz}{dx}\frac{dx}{d\theta}$$

$$=\frac{dz}{dx}\left(-2\left(\frac{n}{10}\right)^{1/6}\sin\theta\right), \tag{4.5}$$

so

$$\frac{d^2u}{d\theta^2}=\frac{d^2z}{dx^2}\left(4\left(\frac{n}{10}\right)^{1/3}\sin^2\theta\right)+\frac{dz}{dx}\left[-2\left(\frac{n}{10}\right)^{1/6}\cos\theta\right]. \tag{4.6}$$

It follows from (4.5) that

$$\frac{dz}{dx}=\frac{du}{d\theta}\left[-\left(\frac{32n}{5}\right)^{-1/6}(\sin\theta)^{-1}\right]. \tag{4.7}$$

Now applying (4.6) and (4.7), we obtain

$$\frac{d^2z}{dx^2}\left(4\left(\frac{n}{10}\right)^{1/3}\sin^2\theta\right)$$

$$=\frac{d^2u}{d\theta^2}-\frac{dz}{dx}\left[-2\left(\frac{n}{10}\right)^{1/6}\cos\theta\right]$$

$$=\frac{d^2u}{d\theta^2}-\frac{du}{d\theta}\left[-\left(\frac{32n}{5}\right)^{-1/6}(\sin\theta)^{-1}\right]\left[-2\left(\frac{n}{10}\right)^{1/6}\cos\theta\right]$$

$$=\frac{d^2u}{d\theta^2}-\frac{du}{d\theta}\cot\theta.$$

Therefore

$$\frac{d^2z}{dx^2}=\left(\frac{d^2u}{d\theta^2}-\frac{du}{d\theta}\cot\theta\right)\left(\frac{1}{4}\left(\frac{n}{10}\right)^{-1/3}\frac{1}{\sin^2\theta}\right). \tag{4.8}$$

Inserting (4.8) into (4.1), we get

$$\left(\frac{d^2u}{d\theta^2}-\frac{du}{d\theta}\cot\theta\right)\left(\frac{1}{4}\left(\frac{n}{10}\right)^{-1/3}\frac{1}{\sin^2\theta}\right)+f_n(x)\,u(\theta)=0$$

or

$$\frac{d^2u}{d\theta^2}-\frac{du}{d\theta}\cot\theta+\left(\frac{32n}{5}\right)^{1/3}\sin^2\theta f_n(x)\,u(\theta)=0. \tag{4.9}$$

Stage 2. Let

$$v(\theta)=\mu(\theta)\left[g(\theta)+\frac{1}{2n}\right]^{1/2}(\sin\theta)^{-1/2},$$
$$\text{wher } g(\theta)=-\frac{1}{10}\cos 6\theta-\frac{2}{5}\cos 4\theta-\frac{1}{2}\cos 2\theta+1 \tag{4.10}$$

and

$$\tau(\theta)=\int_{\pi/2}^{\theta}\left[g(t)+\frac{1}{2n}\right]dt. \tag{4.11}$$

In this stage, we will transform (4.9) to a differential equation in terms of v and τ.

From (4.10), we have

$$u(\theta)=v(\theta)\left[g(\theta)+\frac{1}{2n}\right]^{-1/2}(\sin\theta)^{1/2}$$
$$=v(\theta)(\tau_\theta)^{-1/2}(\sin\theta)^{1/2}. \tag{4.12}$$

Differentiating (4.12) yields

$$u_\theta=v_\tau\tau_\theta(\tau_\theta)^{-1/2}+v[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_\theta$$
$$=v_\tau(\tau_\theta)^{1/2}(\sin\theta)^{1/2}+v[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_\theta \tag{4.13}$$

This implies

$$u_{\theta\Theta}=v_{\tau\tau}(\tau_\theta)^{3/2}(\sin\theta)^{1/2}+v_\tau[\tau_\theta^{1/2}(\sin\theta)^{1/2}]_\theta$$
$$+v_\tau\tau_\theta[(\tau_\theta)^{-1/2}(\sin\theta)^{1/2}]_\theta+v[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_{\theta\theta}$$
$$=v_{\tau\tau}(\tau_\theta)^{3/2}(\sin\theta)^{1/2}+v_\tau[\tfrac{1}{2}\tau_\theta^{-1/2}\tau_{\theta\theta}(\sin\theta)^{1/2}$$
$$+\tfrac{1}{2}\tau_\theta^{1/2}(\sin\theta)^{-1/2}\cos\theta-\tfrac{1}{2}\tau_\theta^{-1/2}\tau_{\theta\theta}(\sin\theta)^{1/2}$$
$$+\tfrac{1}{2}\tau_\theta^{1/2}(\sin\theta)^{-1/2}\cos\theta]+v[(\sin\theta)^{1/2}\tau_\theta^{-1/2}]_{\theta\theta}$$
$$=v_{\tau\tau}(\tau_\theta)^{3/2}(\sin\theta)^{1/2}+v_\tau\tau_\theta^{1/2}(\sin\theta)^{-1/2}\cos\theta$$
$$+v[(\sin\theta)^{1/2}\tau_\theta^{-1/2}]_{\theta\theta}. \tag{4.14}$$

From (4.13), we have

$$u_\theta\cot\theta=v_\tau(\tau_\theta)^{1/2}(\sin\theta)^{-1/2}\cos\theta+v\cot\theta[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_\theta. \tag{4.15}$$

Inserting (4.12), (4.14) and (4.15) into (4.9), we obtain

$$\begin{aligned} v_{\tau\tau}(\tau_\theta)^{3/2}(\sin\theta)^{1/2} &+ v[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_{\theta\theta} \\ &- v\cot\theta[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_\theta \\ &+ v\left(\frac{32n}{5}\right)^{1/3}\sin^2\theta f_n(x)(\tau_\theta)^{-1/2}(\sin\theta)^{1/2} = 0. \quad (4.16) \end{aligned}$$

Applying Lemma 8 to the last term on the left of (4.16), we get

$$\begin{aligned} v_{\tau\tau}(\tau_\theta)^{3/2}(\sin\theta)^{1/2} &+ v[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_{\theta\theta} \\ &- v\cot\theta[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_\theta \\ &+ n^2 v(\tau_\theta)^{3/2}(\sin\theta)^{1/2} + O(1)v = 0, \qquad (4.17) \end{aligned}$$

where we have used the fact that τ_θ is bounded, which can be easily seen from (4.11) and the definition of $g(\theta)$. Also note that from (3.79), if $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$, then

$$\begin{aligned} g(\theta) &= -\frac{1}{10}\cos 6\theta - \frac{2}{5}\cos 4\theta - \frac{1}{2}\cos 2\theta + 1 \\ &= \frac{1}{5}\sin^2\theta(16\cos^4\theta + 8\cos^2\theta + 6) \\ &\geqslant \frac{6}{5}\sin^2\varepsilon > 0, \qquad (4.18) \end{aligned}$$

so

$$\begin{aligned} \tau_\theta &= g(\theta) + \frac{1}{2n} \\ &\geqslant \frac{6}{5}\sin^2\varepsilon > 0. \qquad (4.19) \end{aligned}$$

Hence we see that $(\tau_\theta)^{3/2}(\sin\theta)^{1/2}$ is bounded away from 0. Therefore we can divide (4.17) by $(\tau_\theta)^{3/2}(\sin\theta)^{1/2}$ and obtain

$$\begin{aligned} v_{\tau\tau} &+ \frac{v[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_{\theta\theta}}{(\tau_\theta)^{3/2}(\sin\theta)^{1/2}} \\ &- \frac{v\cot\theta[(\sin\theta)^{1/2}(\tau_\theta)^{-1/2}]_\theta}{(\tau_\theta)^{3/2}(\sin\theta)^{1/2}} + n^2 v = O(1)v. \end{aligned}$$

But in view of (4.18) and (4.19), the second term and the third term of the preceding equation are both $O(1)v$, hence can be absorbed to the right-hand side. Thus, we get

$$v_{\tau\tau} + n^2 v = O(1)v \qquad \text{uniformly for } \varepsilon \leqslant \theta \leqslant \pi - \varepsilon, \frac{\pi}{2} > \varepsilon > 0. \tag{4.20}$$

Stage 3. In this stage, we will solve (4.20) and then get an asymptotic for $p_n(x)$. From Lemma 5, we know that

$$\exp(-x^6/12)\,|p_n(x)| = O(n^{-1/12}) \qquad \text{uniformly for } \varepsilon \leqslant \theta \leqslant \pi - \varepsilon.$$

Thus, from (4.14), we get

$$\begin{aligned} u(\theta) &= z\left(2\left(\frac{n}{10}\right)^{1/6}\cos\theta\right) \\ &= O(n^{-1/12})\,\phi_n^{-1/2}(x) \\ &= O(n^{-1/12})\,O(n^{-1/3}), \end{aligned} \tag{4.21}$$

where, the last equality of (4.21) comes from (3.72). Hence, from (4.10), we have

$$|v(\tau)| = O(n^{-5/12}).$$

Inserting this into (4.20) gives

$$v_{\tau\tau} + n^2 v = O(n^{-5/12}).$$

Solving this as a nonhomogeneous second order differential equation, we obtain

$$\begin{aligned} v(\tau) &= v(0)\cos n\tau + \frac{v_\tau(0)}{n}\sin n\tau \\ &\quad + \int_0^\tau \frac{\cos\tau\cdot\sin nt - \sin n\tau\cdot\cos nt}{-n\sin^2 nt - n\cos^2 nt}\,O(n^{-5/12})\,dt \\ &= v(0)\cos n\tau + \frac{v_\tau(0)}{n}\sin n\tau + \frac{1}{n}\int_0^\tau O(n^{-5/12})\sin[n(\tau - t)]\,dt \\ &= v(0)\cos n\tau + \frac{v_\tau(0)}{n}\sin n\tau + O(n^{-17/12}) \end{aligned} \tag{4.22}$$

uniformly for $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$.

Now, we will find $v(0)$ and $v_\tau(0)/n$ of the preceding equation. If $\tau = 0$, then from (4.11), $\theta = \pi/2$. Consequently $x = 0$ by (4.3). Hence from (4.10) and (4.4), we have the following

$$\begin{aligned} v(0) &= p_n(0)[\phi_n(0)]^{-1/2}\left[g\left(\frac{\pi}{2}\right) + \frac{1}{2n}\right]^{1/2} \\ &= p_n(0)[\phi_n(0)]^{-1/2}\left[\frac{6}{5} + \frac{1}{2n}\right]^{1/2}. \end{aligned} \tag{4.23}$$

By using Lemma 3, (4.23) becomes

$$\begin{aligned} v(0) &= A\cos\left(\frac{n\pi}{2}\right) n^{-1/12}\left(1 + O\left(\frac{1}{n}\right)\right)[\phi_n(0)]^{-1/2}\left(\frac{6}{5}\right)^{1/2}\left(1 + O\left(\frac{1}{n}\right)\right) \\ &= A\cos\left(\frac{n\pi}{2}\right) n^{-1/12}[\phi_n(0)]^{-1/2}\left(\frac{6}{5}\right)^{1/2} + O(n^{-17/12}), \end{aligned} \tag{4.24}$$

where we have used the fact, due to (3.72),

$$[\phi_n(0)]^{-1/2} = O(n^{-1/3}). \tag{4.25}$$

On the other hand, since from (4.10),

$$\begin{aligned} \frac{dv}{d\tau} = \frac{dv}{d\theta}\frac{d\theta}{d\tau} &= [u'(\theta)(\tau_\theta)^{1/2}(\sin\theta)^{-1/2}](\tau_\theta)^{-1} \\ &\quad + \left[u(\theta)\frac{1}{2}(\tau_\theta)^{-1/2}\tau_{\theta\theta}(\sin\theta)^{-1/2}\cdot(\tau_\theta)^{-1}\right] \\ &\quad + \left[u(\theta)(\tau_\theta)^{1/2}\left(-\frac{1}{2}\right)(\sin\theta)^{-3/2}\cos\theta(\tau_\theta)^{-1}\right], \end{aligned}$$

if $\tau = 0$, i.e., $\theta = \pi/2$ or $x = 0$, then

$$v_\tau(0) = u'\left(\frac{\pi}{2}\right)\cdot\left[\tau_\theta\left(\frac{\pi}{2}\right)\right]^{-1/2}. \tag{4.26}$$

Since from (4.5),

$$\begin{aligned} \frac{du}{d\theta} &= \frac{dz}{dx}\left(-2\left(\frac{n}{10}\right)^{1/6}\sin\theta\right) \\ &= \Big[p_n'(x)\,\phi_n^{-1/2}(x)\exp(-x^6/12) \end{aligned}$$

$$+ p_n(x)\left(-\frac{1}{2}\right)\phi_n^{-3/2}(x)\,\phi_n'(x)\exp(-x^6/12)$$

$$+ p_n(x)\,\phi_n^{-1/2}(x)\left(-\frac{1}{2}x^5\right)\exp(-x^6/12)\Bigg]\left[-2\left(\frac{n}{10}\right)^{1/6}\sin\theta\right],$$

(4.26) can be evaluated as

$$v_\tau(0) = p_n'(0)\,\phi_n^{-1/2}(0)\left[-\left(\frac{32n}{5}\right)^{1/6}\right]\left[\frac{6}{5}+\frac{1}{2n}\right]^{-1/2}. \tag{4.27}$$

By applying Lemma 3, (4.27) becomes

$$\begin{aligned} v_\tau(0) &= 6\cdot 10^{-5/6}A\cdot n^{3/4}\sin\left(\frac{n\pi}{2}\right)\left(1+O\left(\frac{1}{n}\right)\right)\phi_n^{-1/2}(0) \\ &\quad \times\left[-\left(\frac{32n}{5}\right)^{1/6}\right]\left(\frac{6}{5}+\frac{1}{2n}\right)^{-1/2} \\ &= -\left(\frac{6}{5}\right)^{1/2}\cdot A\phi_n^{-1/2}(0)\sin\left(\frac{n\pi}{2}\right)n^{11/12}\left(1+O\left(\frac{1}{n}\right)\right), \end{aligned}$$

where A is a constant. Using (4.25), we obtain

$$\frac{v_\tau(0)}{n} = -A\left(\frac{6}{5}\right)^{1/2}[\phi_n(0)]^{-1/2}\sin\left(\frac{n\pi}{2}\right)\cdot n^{-1/12} + O(n^{-17/12}). \tag{4.28}$$

Inserting (4.24) and (4.28) into (4.22), we obtain

$$\begin{aligned} v(\tau) &= A\cos\left(\frac{n\pi}{2}\right)n^{-1/12}[\phi_n(0)]^{-1/2}\left(\frac{6}{5}\right)^{1/2}\cos n\tau \\ &\quad - A\left(\frac{6}{5}\right)^{1/2}[\phi_n(0)]^{-1/2} \\ &\quad \cdot\sin\left(\frac{n\pi}{2}\right)n^{-1/12}\cdot\sin n\tau + O(n^{-17/12}) \\ &= A\left(\frac{6}{5}\right)^{1/2}[\phi_n(0)]^{-1/2}\cdot n^{-1/12}\cos\left(n\tau+\frac{n\pi}{2}\right) + O(n^{-17/12}) \end{aligned} \tag{4.29}$$

uniformly for $\varepsilon \leqslant \theta \leqslant \pi - \varepsilon$.

We can write this equation in terms of $p_n(x)$ as

$$p_n(x)\exp(-x^6/12)\,\phi_n^{-1/2}\left[g(\theta)+\frac{1}{2n}\right]^{1/2}(\sin\theta)^{-1/2}$$
$$=A\left(\frac{6}{5}\right)^{1/2}[\phi_n(0)]^{-1/2}n^{-1/12}\cdot\cos\left(n\tau+\frac{n\pi}{2}\right)+O(n^{-17/12}).$$

Dividing both sides by $\phi_n^{-1/2}(x)[g(\theta)+1/2n]^{1/2}(\sin\theta)^{-1/2}$, and using the fact that $\phi_n^{-1/2}(x)=O(n^{-1/3})$, we obtain

$$p_n(x)\exp(-x^6/12)$$
$$=A\left[\frac{\phi_n(x)}{\phi_n(0)}\right]^{1/2}(\sin\theta)\left[g(\theta)+\frac{1}{2n}\right]^{-1/2}\left(\frac{6}{5}\right)^{1/2}n^{-1/12}$$
$$\cdot(\sin\theta)^{-1/2}\cos\left(n\tau+\frac{n\pi}{2}\right)+O(n^{-13/12}). \tag{4.30}$$

Now consider the expression $[\phi_n(x)/\phi_n(0)]^{1/2}(\sin\theta)[g(\theta)+1/2n]^{-1/2}(6/5)^{1/2}$ in (4.30). Using (3.72) and (3.79) we have

$$\left[\frac{\phi_n(x)}{\phi_n(0)}\right]^{1/2}(\sin\theta)\left[g(\theta)+\frac{1}{2n}\right]^{1/2}\left(\frac{6}{5}\right)^{1/2}$$
$$=\left[\frac{(n/10)^{2/3}(8\cos^2\theta+16\cos^4\theta+6)+O(n^{-1/3})}{2(n/10)^{2/3}(3+O(1/n))}\right]^{1/2}$$
$$\cdot\sin\theta\cdot\frac{(6/5)^{1/2}}{[1/5\sin^2\theta(16\cos^4\theta+8\cos^2\theta+6)+1/2n]^{1/2}}$$
$$=\left\{\left(\frac{n}{10}\right)^{2/3}\left[\frac{1}{6}\left(\frac{n}{10}\right)^{-2/3}\right](8\cos^2\theta+16\cos^4\theta+6)+O(n^{-1})\right\}^{1/2}$$
$$\cdot\{6^{1/2}(16\cos^4\theta+8\cos^2\theta+6)^{-1/2}+O(n^{-1})\}$$
$$=\left[\left(\frac{1}{6}\right)^{1/2}(8\cos^2\theta+16\cos^4\theta+6)^{1/2}\right.$$
$$\left.+O\left(\frac{1}{n}\right)\right][\sqrt{6}(16\cos^4\theta+8\cos^2\theta+6)^{-1/2}+O(n^{-1})]$$
$$=1+O(n^{-1}).$$

Inserting this into (4.30), then we have

$$p_n(x)\exp(-x^6/12) = An^{-1/12}(\sin\theta)^{-1/2}\cos\left(n\tau+\frac{n\pi}{2}\right)+O(n^{-13/12}). \qquad (4.31)$$

Stage 4. In this stage, we are going to find the constant A and complete the proof. Before we proceed, we make the following remark: If $h(\theta)$ is a continuous function then by a change of variable, we have

$$\int_{\varepsilon}^{\pi-\varepsilon} h(\theta)\cos(2n\tau+n\pi)\,d\theta = (-1)^n\int_{\varepsilon_1}^{\varepsilon_2} h(\theta)[g(\theta)]^{-1}(\sin\theta\cos 2nw+\cos\theta\sin 2nw)\,dw, \qquad (4.32)$$

where $w=\tau-(\theta-\pi/2)/2n$,

$$\varepsilon_1=\int_{\pi/2}^{\varepsilon} g(t)\,dt,$$

and

$$\varepsilon_2=\int_{\pi/2}^{\pi-\varepsilon} g(t)\,dt.$$

Note that from (4.19), w is a continuous, strictly increasing function of θ if n is sufficiently large, so its inverse exists. Hence in (4.32) we can also express $h(\theta)[g(\theta)]^{-1}\sin\theta$ or $h(\theta)[g(\theta)]^{-1}\cos\theta$ as a continuous function of w. Thus, we can apply the Riemann–Lebesgue lemma to conclude that

$$\lim_{n\to\infty}\int_{\varepsilon}^{\pi-\varepsilon} h(\theta)\cos(2n\tau+n\pi)\,d\theta=0. \qquad (4.33)$$

Now, let us return to our problem.
Since

$$\int_{|x|\leqslant(32n/5)^{1/6}\cos\varepsilon} p_n^2(x)\exp(-x^6/6)\,dx \leqslant \int_{-\infty}^{\infty} p_n^2(x)\exp(-x^6/6)\,dx=1$$

it follows from (4.31) that

$$\int_{\pi-\varepsilon}^{\varepsilon}\left\{\left[A^2\cdot n^{-1/6}(\sin\theta)^{-1}\cos^2\left(n\tau+\frac{n\pi}{2}\right)\right]\right.$$
$$+2\left[An^{-1/12}(\sin\theta)^{-1/2}\cdot\cos\left(n\tau+\frac{n\pi}{2}\right)\cdot O(n^{-13/12})\right]$$
$$\left.+O(n^{-13/6})\right\}\left[-\left(\frac{32n}{5}\right)^{1/6}\sin\theta\right]d\theta\leqslant 1, \tag{4.34}$$

or

$$\int_{\varepsilon}^{\pi-\varepsilon}A^2\left(\frac{32}{5}\right)^{1/6}\cos^2\left(n\tau+\frac{n\pi}{2}\right)d\theta\leqslant 1+O\left(\frac{1}{n}\right). \tag{4.35}$$

Since

$$\cos^2\left(n\tau+\frac{n\pi}{2}\right)=\frac{1+\cos(2n\tau+n\pi)}{2},$$

(4.35) can be written as

$$A^2(\pi-2\varepsilon)\left(\frac{32}{5}\right)^{1/6}\cdot\frac{1}{2}+A^2\left(\frac{32}{5}\right)^{1/6}$$
$$\cdot\frac{1}{2}\int_{\varepsilon}^{\pi-\varepsilon}\cos(2n\tau+n\pi)\,d\theta\leqslant 1+O\left(\frac{1}{n}\right).$$

Letting $n\to\infty$ and applying (4.33), we obtain

$$A^2(\pi-2\varepsilon)\left(\frac{1}{10}\right)^{1/6}\leqslant 1.$$

Since $0<\varepsilon<\pi/2$ is arbitrary,

$$A^2\cdot\pi\cdot\left(\frac{1}{10}\right)^{1/6}\leqslant 1.$$

So

$$A\leqslant 10^{1/12}\cdot\pi^{-1/2}.$$

Next, we will show $A\geqslant 10^{1/12}\pi^{-1/2}$ to conclude that $A=10^{1/12}\pi^{-1/2}$. Applying the recursion formula, we obtain

$$\int_{-\infty}^{\infty}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right]p_n^2(x)\exp(-x^6/6)\,dx$$
$$=\int_{-\infty}^{\infty}p_n^2(x)\exp(-x^6/6)\,dx$$
$$-\int_{-\infty}^{\infty}\left(\frac{32n}{5}\right)^{-1/3}x^2p_n^2(x)\exp(-x^6/6)\,dx$$
$$=1-\left(\frac{32n}{5}\right)^{-1/3}(a_{n+1}^2+a_n^2). \tag{4.36}$$

Since if $|x|>(32n/5)^{1/6}$, then $1-x^2(32n/5)^{-1/3}<0$, we have

$$\int_{-\infty}^{\infty}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right]p_n^2(x)\exp(-x^6/6)\,dx$$
$$\leqslant\int_{|x|\leqslant(32n/5)^{1/6}}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right]p_n^2(x)\exp(-x^6/6)\,dx. \tag{4.37}$$

Combining (4.36) and (4.37), we see that

$$1-\left(\frac{32n}{5}\right)^{1/3}(a_{n+1}^2+a_n^2)$$
$$\leqslant\int_{|x|\leqslant(32n/5)^{1/6}}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right]p_n^2(x)\exp(-x^6/6)\,dx$$
$$\leqslant\int_{|x|\leqslant(32n/5)^{1/6}\cos\varepsilon}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right]p_n^2(x)\exp(-x^6/6)\,dx$$
$$+\int_{\cos\varepsilon(32n/5)^{1/6}\leqslant|x|\leqslant(32n/5)^{1/6}}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right]$$
$$\times\, p_n^2(x)\exp(-x^6/6)\,dx. \tag{4.38}$$

From (4.3) and the asymptotic expression (4.31), it follows that

$$\int_{|x|\leqslant(32n/5)^{1/6}\cos\varepsilon}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right]p_n^2(x)\exp(-x^6/6)\,dx$$
$$=\int_{\varepsilon}^{\pi-\varepsilon}\sin^2\theta\left\{\left[A^2n^{-1/6}(\sin\theta)^{-1}\cos^2\left(n\tau+\frac{n\pi}{2}\right)\right]\right.$$
$$+2\left[An^{-1/2}(\sin\theta)^{-1/2}\cdot\cos\left(n\tau+\frac{n\pi}{2}\right)O(n^{-13/12})\right]$$

$$+ O(n^{-13/6})\Big\}\left[\left(\frac{32n}{5}\right)^{1/6}\sin\theta\right] d\theta$$

$$= \int_{\varepsilon}^{\pi-\varepsilon} \sin^2\theta \left[A^2 n^{-1/6}(\sin\theta)^{-1}\cos^2\left(n\tau+\frac{n\pi}{2}\right)\right]$$

$$\cdot\left[\left(\frac{32n}{5}\right)^{1/6}\sin\theta\right] d\theta + O(n^{-1})$$

$$= A^2\left(\frac{1}{10}\right)^{1/6}\int_{\varepsilon}^{\pi-\epsilon}\sin^2\theta\, d\theta$$

$$+ A^2\left(\frac{1}{10}\right)^{1/6}\int_{\varepsilon}^{\pi-\varepsilon}\sin^2\theta\cos(2n\tau+n\pi)\, d\theta + O(n^{-1}). \quad (4.39)$$

Now let $n\to\infty$ and apply (4.33), we get

$$\limsup_{n\to\infty}\int_{|x|\leqslant\cos\varepsilon(32n/5)^{1/6}}\left[1-x^2\left(\frac{32n}{5}\right)^{-1/3}\right] p_n^2(x)\exp(-x^6/6)\, dx$$

$$\leqslant A^2\left(\frac{1}{10}\right)^{1/6}\frac{\pi}{2}. \quad (4.40)$$

Combining (4.38), (4.40) and Lemma 7, we see that if $n\to\infty$ then

$$\frac{1}{2}\leqslant A^2\left(\frac{1}{10}\right)^{1/6}\cdot\frac{\pi}{2}+c(1-\cos\varepsilon).$$

Let $\varepsilon\to 0$, then

$$\frac{1}{2}\leqslant A^2\left(\frac{1}{10}\right)^{1/6}\frac{\pi}{2},$$

or

$$A^2\geqslant 10^{1/6}\pi^{-1}.$$

Since $A>0$ by Lemma 3, we conclude that

$$A\geqslant 10^{1/12}\pi^{-1/2}.$$

Thus

$$A = 10^{1/12}\pi^{-1/2}.$$

Replacing A by $10^{1/12}\pi^{-1/2}$ in (4.31), we obtain

$$p_n(x)\exp(-x^6/12)$$
$$= 10^{1/12}\pi^{-1/2}n^{-1/12}(\sin\theta)^{-1/2}\cos\left(n\tau+\frac{n\pi}{2}\right)+O(n^{-13/12}),$$

where

$$\tau=\int_{\pi/2}^{\theta}\left[g(t)+\frac{1}{2n}\right]dt$$
$$=\int_{\pi/2}^{\theta}\left[-\frac{1}{10}\cos 6t-\frac{2}{5}\cos 4t-\frac{1}{2}\cos 2t+1+\frac{1}{2n}\right]dt$$
$$=-\frac{1}{60}\sin 6\theta-\frac{1}{10}\sin 4\theta-\frac{1}{4}\sin 2\theta+\theta-\frac{\pi}{2}+\frac{\theta}{2n}-\frac{\pi}{4n}.$$

Consequently,

$$p_n(x)\exp(-x^6/12)$$
$$= 10^{1/12}\pi^{-1/2}n^{-1/12}(\sin\theta)^{-1/2}\cos\left[\frac{n}{60}(60\theta-15\sin 2\theta-6\sin 4\theta\right.$$
$$\left.-\sin 6\theta)+\frac{\theta}{2}-\frac{\pi}{4}\right]+O(n^{-13/12}).$$

The proof of this theorem is complete.

Proof of Theorem 2. From (3.38) we have

$$\sum_{k=0}^{k-1} p_k^2(x)=a_n^2\phi_{n-1}(x)\,p_n^2(x)+a_n^2\phi_n(x)\,p_{n-1}^2(x)$$
$$+a_n[\pi_{n-1}(x)-\pi_n(x)-x\phi_{n-1}(x)]\,p_n(x)\,p_{n-1}(x). \quad (4.41)$$

Applying (3.71) and (3.72), we obtain

$$\sum_{k=0}^{n-1} p_k^2(x)=\left[\left(\frac{n}{10}\right)^{1/3}\left(1+O\left(\frac{1}{n^2}\right)\right)\right]\left[\left(\frac{n}{10}\right)^{2/3}B+O(n^{-1/3})\right]p_n^2(x)$$
$$+\left[\left(\frac{n}{10}\right)^{1/3}\left(1+O\left(\frac{1}{n^2}\right)\right)\right]$$
$$\cdot\left[\left(\frac{n}{10}\right)^{2/3}B+O(n^{-1/3})\right]p_{n-1}^2(x)$$

$$-\left(\frac{n}{10}\right)^{1/6}\left(1+O\left(\frac{1}{n^2}\right)\right)\left\{O(n^{-1/6})\right.$$

$$\left.+\left(\frac{32n}{5}\right)^{1/6}\cos\theta\cdot\left[\left(\frac{n}{10}\right)^{2/3}B+O(n^{-1/3})\right]\right\}p_n(x)\,p_{n-1}(x),$$

where $B=16\cos^4\theta+8\cos^2\theta+6$. Simplifying, we get

$$\sum_{k=0}^{n-1}p_k^2(x)=\left[\left(\frac{n}{10}\right)B+O(1)\right]p_n^2(x)+\left[\left(\frac{n}{10}\right)B+O(1)\right]p_{n-1}^2(x)$$

$$-2\left[\left(\frac{n}{10}\right)B\cos\theta+O(1)\right]p_n(x)\,p_{n-1}(x)$$

$$=\left(\frac{n}{10}\right)Bp_n^2(x)+O(n^{-1/6})\exp(x^6/6)$$

$$+\left(\frac{n}{10}\right)B\cdot p_{n-1}^2(x)+O(n^{-1/6})\exp(x^6/6)$$

$$-2\left(\frac{n}{10}\right)B\cos\theta p_n(x)\,p_{n-1}(x)+O(n^{-1/6})\exp(x^6/6)$$

$$=\frac{n}{10}(16\cos^4\theta+8\cos^2\theta+6)[p_n^2(x)+p_{n-1}^2(x)$$

$$-2\cos\theta p_n(x)\,p_{n-1}(x)]+O(n^{-1/6})\exp(x^6/6). \qquad (4.42)$$

Multiplying (4.42) by $\exp(-x^6/6)$ gives

$$\exp(-x^6/6)\sum_{k=0}^{n-1}p_k^2(x)$$

$$=\frac{n}{10}(16\cos^4\theta+8\cos^2\theta+6)[p_n^2(x)+p_{n-1}^2(x)$$

$$-2\cos\theta p_n(x)\,p_{n-1}(x)]\cdot\exp(-x^6/6)+O(n^{-1/6}). \qquad (4.43)$$

Thus, if we can find the asymptotic for $[p_n^2(x)+p_{n-1}^2(x)-2\cos\theta p_n(x)\,p_{n-1}(x)]\cdot\exp(-x^6/6)$, then the theorem will be proved easily. Applying Theorem 1,

$$p_n(x)\exp(-x^6/12)(\sin\theta_1)^{1/2}$$

$$=10^{1/12}\pi^{-1/2}n^{-1/12}\cos\left(n\tau_1+\frac{n\pi}{2}\right)+O(n^{-13/12}). \qquad (4.44)$$

Squaring (4.44), we have

$$p_n^2 \exp(-x^6/6) = \frac{10^{1/6}\pi^{-1}n^{-1/6}\cos^2(n\tau_1 + n\pi/2)}{\sin\theta_1} + O(n^{-7/6}), \quad (4.45)$$

where $x = (32n/5)^{1/6}\cos\theta_1$ and $\tau_1 = \int_{\pi/2}^{\theta_1}[g(t) + 1/2n]\,dt$, $g(t)$ is the function defined in Lemma 8.

Also, using Theorem 1, we have

$$p_{n-1}(x)\exp(-x^6/12)$$
$$= 10^{1/12}\pi^{-1/2}n^{-1/12}(\sin\theta_2)^{-1/2}\cos\left[(n-1)\tau_2 + \frac{(n-1)\pi}{2}\right] + O(n^{-13/12}). \quad (4.46)$$

Squaring both sides of (4.46), we obtain

$$p_{n-1}^2(x)\exp(-x^6/6)\sin\theta_2$$
$$= 10^{1/6}\pi^{-1}n^{-1/6}\cos^2\left[(n-1)\tau_2 + \frac{(n-1)\pi}{2}\right] + O(n^{-7/6}), \quad (4.47)$$

where $x = [32(n-1)/5]^{1/6}\cos\theta_2$ and $\tau_2 = \int_{\pi/2}^{\theta_2}[g(t) + 1/2(n-1)]\,dt$. In the above, we have used θ_1, θ_2, τ_1 and τ_2 to denote the distinct values. In view of (4.44), (4.45), (4.46) and (4.47), to find the asymptotic for $[p_n^2(x) + p_{n-1}^2(x) - 2\cos\theta p_n(x)\,p_{n-1}(x)]\cdot\exp(-x^6/6)$, the main task will be investigating the asymptotic relationship between θ_1 and θ_2 and also between τ_1 and τ_2. Let us consider the former case first. Since $x = (32n/5)^{1/6}\cos\theta_1 = (32(n-1)/5)^{1/6}\cos\theta_2$ we have

$$\left(\frac{32n}{5}\right)^{1/6}\cos\theta_1 = \left(\frac{32n}{5}\right)^{1/6}\left(1 - \frac{1}{n}\right)^{1/6}\cos\theta_2.$$

So

$$\cos\theta_1 = \left(1 - \frac{1}{n}\right)^{1/6}\cos\theta_2, \quad (4.48)$$

or

$$\theta_2 = \cos^{-1}\left[\frac{\cos\theta_1}{(1 - 1/n)^{1/6}}\right]. \quad (4.49)$$

From the above derivation and (4.49), we see that given θ_1, and if n is sufficiently large, then we always can find θ_2 such that $(32n/5)^{1/6}\cos\theta_1 = (32(n-1)/5)^{1/6}\cos\theta_2$, i.e., they represent same value for x. Also observe from (4.49) that if n increases, then θ_2 approaches θ_1 from the right if θ_1 is in the second quadrant and from left if θ_1 is in the first quadrant. This observation leads to the following remark: If $0<\varepsilon_1<\pi/2$ and $x=(32n/5)^{1/6}\cos\theta_1$, where $\varepsilon_1\leqslant\theta_1\leqslant\pi-\varepsilon_1$, then there exists n_1 and ε_2 depending on ε_1 such that $0<\varepsilon_2<\pi/2$ and if $x=(32(n-1)/5)^{1/6}\cos\theta_2$ and $n>n_1$ then $\varepsilon_2\leqslant\theta_2\leqslant\pi-\varepsilon_2$. This fact makes the operations on (4.46) and (4.47) possible as $n\to\infty$. From (4.48), we obtain

$$\theta_1=\cos^{-1}\left[\left(1-\frac{1}{n}\right)^{1/6}\cos\theta_2\right]$$

$$=\cos^{-1}\left[\cos\theta_2-\frac{\cos\theta_2}{6n}+O\left(\frac{1}{n^2}\right)\right]. \tag{4.50}$$

Since the derivative of $\cos^{-1}(x)$ is $-1/\sin[\cos^{-1}(x)]$, applying the Mean Value Theorem to (4.50), we get (for the time being, fix n)

$$\theta_1=\cos^{-1}(\cos\theta_2)+\frac{-1}{\sin[\cos^{-1}(\cos\theta_0)]}\left[\frac{-\cos\theta_2}{6n}+O\left(\frac{1}{n^2}\right)\right]$$

$$=\theta_2+\frac{-1}{\sin\theta_0}\left[\frac{-\cos\theta_2}{6n}+O\left(\frac{1}{n^2}\right)\right], \tag{4.51}$$

where $\cos\theta_0$ is a value between $\cos\theta_2$ and $\cos\theta_2-\cos\theta_2/6n+O(1/n^2)$. Since the asymptotic for θ_1 is unique, and $\theta_0\to\theta_2$ as $n\to\infty$, we obtain from (4.51)

$$\theta_1=\theta_2+\frac{1}{6n}\cot\theta_2+O\left(\frac{1}{n^2}\right). \tag{4.52}$$

Next, let us investigate the relation between τ_1 and τ_2. By the above definition of τ_1 and τ_2, we have

$$\tau_1=\int_{\pi/2}^{\theta_1}\left[g(t)+\frac{1}{2n}\right]dt$$

$$=\int_{\pi/2}^{\theta_1}g(t)\,dt+\int_{\pi/2}^{\theta_1}\left(\frac{1}{2n}\right)dt$$

$$=\int_{\pi/2}^{\theta_2}g(t)\,dt+\int_{\theta_2}^{\theta_1}g(t)\,dt+\int_{\pi/2}^{\theta_1}\frac{1}{2n}\,dt, \tag{4.53}$$

and

$$\tau_2 = \int_{\pi/2}^{\theta_2} \left[g(t) + \frac{1}{2(n-1)} \right] dt$$

$$= \int_{\pi/2}^{\theta_2} g(t)\, dt + \int_{\pi/2}^{\theta_2} [2(n-1)]^{-1}\, dt. \tag{4.54}$$

Thus, with the aid of (4.52), we obtain

$$\tau_1 - \tau_2 = \int_{\theta_2}^{\theta_1} g(t)\, dt + \left(\theta_1 - \frac{\pi}{2}\right)(2n)^{-1} - \left(\theta_2 - \frac{\pi}{2}\right)[2(n-1)]^{-1}$$

$$= \int_{\theta_2}^{\theta_1} g(t)\, dt + \frac{\theta_1}{2n} - \frac{\theta_2}{2}(n^{-1} + O(n^{-2})) - \frac{\pi}{4n} + \frac{\pi}{4(n-1)}$$

$$= \int_{\theta_2}^{\theta_1} g(t)\, dt + (\theta_1 - \theta_2)\frac{1}{2n} + O(n^{-2})$$

$$= \int_{\theta_2}^{\theta_1} g(t)\, dt + O(n^{-2}). \tag{4.55}$$

Now applying the Mean Value Theorem for integrals, we get

$$\tau_1 - \tau_2 = (\theta_1 - \theta_2)\, g(\theta') + O(n^{-2}), \tag{4.56}$$

where θ' is a value between θ_1 and θ_2, so we can write θ' as $\theta' = \theta_2 + k$, where $|k| < |\theta_1 - \theta_2| = O(1/n)$. Therefore, by using the Mean Value Theorem, we obtain from (4.56)

$$\tau_1 - \tau_2 = (\theta_1 - \theta_2)\, g(\theta_2 + k) + O(n^{-2})$$

$$= (\theta_1 - \theta_2)[g(\theta_2) + g'(\zeta)k] + O(n^{-2})$$

$$= (\theta_1 - \theta_2)\, g(\theta_2) + O(n^{-2})$$

$$= \frac{1}{6n} \cot \theta_2\, g(\theta_2) + O(n^{-2}). \tag{4.57}$$

Now return to (4.46) and (4.47). We will investigate $(n-1)\tau_2 + (n-1)\pi/2$ and $\sin \theta_2$ in these two equations. From (4.57), we get

$$n\tau_1 - n\tau_2 = \tfrac{1}{6} \cot \theta_2\, g(\theta_2) + O(n^{-1}).$$

Hence

$$n\tau_1 - n\tau_2 + \tau_2 = \frac{1}{6} \cot \theta_2\, g(\theta_2) + \int_{\pi/2}^{\theta_2} \left[g(t) + \frac{1}{2(n-1)} \right] dt$$

$$= \frac{1}{6} \cot \theta_2\, g(\theta_2) + \int_{\pi/2}^{\theta_2} g(t)\, dt + O\left(\frac{1}{n}\right).$$

Therefore

$$n\tau_1-(n-1)\tau_2+\frac{\pi}{2}=\frac{1}{6}\cot\theta_2\, g(\theta_2)+\int_{\pi/2}^{\theta_2} g(t)\,dt+\frac{\pi}{2}+(n^{-1}). \quad (4.58)$$

Using (3.79), we get

$$\begin{aligned}\frac{1}{6}\cot\theta_2\, g(\theta_2)&=\frac{1}{6}\cot\theta_2\left[\frac{1}{5}\sin^2\theta_2(16\cos^4\theta_2+8\cos^2\theta_2+6)\right]\\ &=\frac{1}{30}\cos\theta_2\sin\theta_2(16\cos^4\theta_2+8\cos^2\theta_2+6)\\ &=\frac{1}{60}\sin 2\theta_2\left[16\left(\frac{1+2\cos 2\theta_2}{4}+\frac{1+\cos 4\theta_2}{8}\right)\right.\\ &\quad\left.+8\left(\frac{1+\cos 2\theta_2}{2}\right)+6\right]\\ &=\frac{1}{60}\sin 2\theta_2[16+12\cos 2\theta_2+2\cos 4\theta_2]\\ &=\frac{16}{60}\sin 2\theta_2+\frac{3}{15}\sin 2\theta_2\cos 2\theta_2+\frac{1}{30}\sin 2\theta_2\cos 4\theta_2\\ &=\frac{1}{4}\sin 2\theta_2+\frac{1}{10}\sin 4\theta_2+\frac{1}{60}\sin 6\theta_2, \end{aligned} \quad (4.59)$$

and

$$\begin{aligned}\int_{\pi/2}^{\theta_2} g(t)\,dt&=\int_{\pi/2}^{\theta_2}\left[-\frac{1}{10}\cos 6t-\frac{2}{5}\cos 4t-\frac{1}{2}\cos 2t+1\right]dt\\ &=-\frac{1}{60}\sin 6\theta_2-\frac{1}{10}\sin 4\theta_2-\frac{1}{4}\sin 2\theta_2+\theta_2-\frac{\pi}{2}. \end{aligned} \quad (4.60)$$

Inserting (4.59) and (4.60) into (4.58), we obtain

$$\begin{aligned}n\tau_1-(n-1)\tau_2+\frac{\pi}{2}&=\theta_2-\frac{\pi}{2}+\frac{\pi}{2}+O(n^{-1})\\ &=\theta_2+O(n^{-1})\\ &=\theta_1+O(n^{-1}),\end{aligned}$$

which can be written as

$$n\tau_1+\frac{n\pi}{2}=(n-1)\tau_2+(n-1)\frac{\pi}{2}+\theta_1+O(n^{-1}). \quad (4.61)$$

Also, from (4.52) and the fact that $\sin\theta_1$ and $\sin\theta_2$ are bounded away from zero, we have

$$\begin{aligned}\frac{1}{\sin\theta_1} &= \frac{1}{\sin(\theta_2 + O(1/n))}\\ &= \frac{1}{\sin\theta_2 + \cos(\theta')\, O(1/n)}\\ &= (\sin\theta_2)^{-1}\left[1 + O\left(\frac{1}{n}\right)\right]^{-1}\\ &= (\sin\theta_2)^{-1} + O\left(\frac{1}{n}\right). \qquad (4.62)\end{aligned}$$

Applying (4.61) and (4.62), we can write (4.46) and (4.47) as

$$\begin{aligned}&p_{n-1}(x)\exp(-x^6/12)(\sin\theta_1)^{1/2}\\ &\quad = 10^{1/12}\pi^{-1/2}n^{-1/12}\cos\left(n\tau_1 + \frac{n\pi}{2} - \theta_1\right) + O(n^{-13/12}), \qquad (4.63)\end{aligned}$$

and

$$\begin{aligned}&p_{n-1}^2(x)\exp(-x^6/6)\\ &\quad = \frac{10^{1/6}\pi^{-1}n^{-1/6}\cos^2(n\tau_1 + n\pi/2 - \theta_1)}{\sin\theta_1} + O(n^{-7/6}). \qquad (4.64)\end{aligned}$$

Now applying (4.44), (4.45), (4.63) and (4.64) to the expression $[p_n^2(x) + p_{n-1}^2(x) - 2\cos\theta_1\, p_n(x)\, p_{n-1}(x)]\cdot\exp(-x^6/6)$ in (4.43), we obtain

$$\begin{aligned}&[p_n^2(x) + p_{n-1}^2(x) - 2\cos\theta_1\, p_n(x)\, p_{n-1}(x)]\cdot\exp(-x^6/6)\\ &\quad = \frac{10^{1/6}\pi^{-1}n^{-1/6}[\cos^2(n\tau_1 + n\pi/2) + \cos^2(n\tau_1 + n\pi/2 - \theta_1)]}{\sin\theta_1}\\ &\qquad - \frac{2\cos\theta_1\cdot 10^{1/6}\pi^{-1}n^{-1/6}\cos(n\tau_1 + n\pi/2)\cdot\cos(n\tau_1 + n\pi/2 - \theta_1)}{\sin\theta_1}\\ &\qquad + O(n^{-7/6})\\ &\quad = \{10^{1/6}\pi^{-1}n^{-1/6}[1 + \cos(2n\tau_1 + n\pi - \theta_1)\cos\theta_1]\\ &\qquad - 10^{1/6}\pi^{-1}n^{-1/6}\cos\theta_1\,[\cos(2n\tau_1 + n\pi - \theta_1)\\ &\qquad + \cos\theta_1]\}/\sin\theta_1\end{aligned}$$

$$\begin{aligned}
&+O(n^{-7/6})\\
&=10^{1/6}\pi^{-1}n^{-1/6}\{[1+\cos(2n\tau_1+n\pi-\theta_1)\cos\theta_1]\\
&\quad-(\cos\theta_1)\cdot[\cos(2n\tau_1+n\pi-\theta_1)\\
&\quad+\cos\theta_1]\}/\sin\theta_1+O(n^{-7/6})\\
&=\frac{10^{1/6}\pi^{-1}n^{-1/6}(1-\cos^2\theta_1)}{\sin\theta_1}+O(n^{-7/6})\\
&=10^{1/6}\pi^{-1}n^{-1/6}\sin\theta_1+O(n^{-7/6}).
\end{aligned}\tag{4.65}$$

Inserting (4.65) into (4.43), we conclude that

$$\begin{aligned}
&\sum_{k=0}^{n-1} p_k^2(x)\exp(-x^6/6)\\
&=\frac{n}{10}(16\cos^4\theta_1+8\cos^2\theta_1+6)[10^{1/6}\pi^{-1}n^{-1/6}\sin\theta_1\\
&\quad+O(n^{-7/6})]+O(n^{-1/6})\\
&=\frac{n}{10}(16\cos^4\theta_1+8\cos^2\theta_1+6)\cdot 10^{1/6}\cdot\pi^{-1}n^{-1/6}\sin\theta_1+O(n^{-1/6})\\
&=10^{-5/6}\pi^{-1}\sin\theta_1(16\cos^4\theta_1+8\cos^2\theta_1+6)\cdot n^{5/6}+O(n^{-1/6}).
\end{aligned}\tag{4.66}$$

Now the theorem follows immediately from (4.66).

Proof of Theorem 3. The proof will be derived from Theorem 1. If $x\in\Delta$, then x can be expressed as $(32n/5)^{1/6}\cos\theta$, so

$$\theta=\cos^{-1}\left[\left(\frac{5}{32n}\right)^{1/6}x\right].\tag{4.67}$$

Based on this equality, we can change all the expressions in Theorem 1 which involve θ to the ones in terms of x. We do this beginning with the expression inside the parenthesis. Applying the equalities

$$\sin 6\theta=4\sin\theta\cdot\cos\theta\cdot\cos^2 2\theta+2\sin\theta\cdot\cos\theta\cdot\cos 4\theta$$

and

$$\cos 4\theta=8\cos^4\theta-8\cos^2\theta+1,$$

we obtain

$$\begin{aligned}
&15\sin 2\theta+6\sin 4\theta+\sin 6\theta\\
&\quad=30\sin\theta\cdot\cos\theta+24\sin\theta\cdot\cos\theta\cdot\cos 2\theta+4\sin\theta\cdot\cos\theta\\
&\qquad\cdot(4\cos^4\theta-4\cos^2\theta+1)+2\sin\theta\cdot\cos\theta\cdot(8\cos^4\theta-8\cos^2\theta+1)\\
&\quad=\sin\theta\cdot(32\cos^5\theta+16\cos^3\theta+12\cos\theta).
\end{aligned}\tag{4.68}$$

In view of (4.67), we can write $\sin\theta$ as follows:

$$\begin{aligned}
\sin\theta&=[1-\cos^2\theta]^{1/2}\\
&=1-\frac{1}{2}\cos^2\theta-\frac{1}{8}\cos^4\theta-\frac{1}{16}\cos^6\theta\\
&\quad-\frac{5}{128}\cos^8\theta-\frac{7}{256}\cos^{10}\theta+O(n^{-2}).
\end{aligned}\tag{4.69}$$

It follows that

$$\begin{aligned}
&15\sin 2\theta+6\sin 4\theta+\sin 6\theta\\
&\quad=\left[1-\frac{1}{2}\cos^2\theta-\frac{1}{8}\cos^4\theta-\frac{1}{16}\cos^6\theta\right.\\
&\qquad\left.-\frac{5}{128}\cos^8\theta-\frac{7}{256}\cos^{10}\theta+O(n^{-2})\right]\\
&\qquad\cdot(32\cos^5\theta+16\cos^3\theta+12\cos\theta)\\
&\quad=12\cos\theta+10\cos^3\theta+\frac{45}{2}\cos^5\theta\\
&\qquad-\frac{75}{4}\cos^7\theta-\frac{175}{32}\cos^9\theta-\frac{189}{64}\cos^{11}\theta+O(n^{-2})\\
&\quad=12\left(\frac{5}{32n}\right)^{1/6}x+10\left(\frac{5}{32n}\right)^{1/2}x^3+\frac{45}{2}\left(\frac{5}{32n}\right)^{5/6}x^5-\frac{75}{4}\left(\frac{5}{32n}\right)^{7/6}x^7\\
&\qquad-\frac{175}{32}\left(\frac{5}{32n}\right)^{9/6}x^9-\frac{189}{64}\left(\frac{5}{32n}\right)^{11/6}x^{11}+O(n^{-2}).
\end{aligned}\tag{4.70}$$

From (4.67),

$$\begin{aligned}\theta &= \cos^{-1}\left[\left(\frac{5}{32n}\right)^{1/6} x\right]\\ &= \frac{\pi}{2} - \left(\frac{5}{32n}\right)^{1/6} x - \frac{1}{6}\left(\frac{5}{32n}\right)^{3/6} x^3\\ &\quad - \frac{3}{40}\left(\frac{5}{32n}\right)^{5/6} x^5 - \frac{5}{112}\left(\frac{5}{32n}\right)^{7/6} x^7\\ &\quad - \frac{35}{1152}\left(\frac{5}{32n}\right)^{9/6} x^9 - \frac{63}{2816}\left(\frac{5}{32n}\right)^{11/6} x^{11} + O(n^{-2}). \qquad (4.71)\end{aligned}$$

Combining (4.70) and (4.71), we have

$$\begin{aligned}&60\theta - 15\sin 2\theta - 6\sin 4\theta - \sin 6\theta\\ &\quad = 30\pi - 72\left(\frac{5}{32n}\right)^{1/6} x - 20\left(\frac{5}{32n}\right)^{3/6} x^3\\ &\quad\quad - 27\left(\frac{5}{32n}\right)^{5/6} x^5 + \frac{225}{14}\left(\frac{5}{32n}\right)^{7/6} x^7\\ &\quad\quad + \frac{175}{48}\left(\frac{5}{32n}\right)^{9/6} x^9 + \frac{567}{352}\left(\frac{5}{32n}\right)^{11/6} x^{11} + O(n^{-2}). \qquad (4.72)\end{aligned}$$

Now the asymptotic of the bracket-expression in Theorem 1 can be obtained by using (4.71) and (4.72) as follows:

$$\begin{aligned}&\frac{n}{60}(60\theta - 15\sin 2\theta - 6\sin 4\theta - \sin 6\theta) + \frac{\theta}{2} - \frac{\pi}{4}\\ &\quad = \frac{\pi}{2} n - \frac{6}{5}\left(\frac{5}{32}\right)^{1/6} \cdot x \cdot n^{5/6} - \frac{1}{3}\left(\frac{5}{32}\right)^{1/2} \cdot x^3 \cdot n^{1/2}\\ &\quad\quad - \frac{9}{20}\left(\frac{5}{32}\right)^{5/6} \cdot x^5 \cdot n^{1/6}\\ &\quad\quad + \left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6} x^7 - \frac{1}{2}\left(\frac{5}{32}\right)^{1/6} x\right] n^{-1/6}\\ &\quad\quad + \left[\frac{35}{576}\left(\frac{5}{32}\right)^{3/2} x^9 - \frac{1}{12}\left(\frac{5}{32}\right)^{1/2} x^3\right] n^{-1/2}\\ &\quad\quad + \left[\frac{189}{7040}\left(\frac{5}{32}\right)^{11/6} x^{11} - \frac{3}{80}\left(\frac{5}{32}\right)^{5/6} x^5\right] n^{-5/6} + O(n^{-1}).\end{aligned}$$

Hence

$$
\begin{aligned}
&\cos\left[\frac{n}{60}(60\theta - 15\sin 2\theta - 6\sin 4\theta - \sin 6\theta) + \frac{\theta}{2} - \frac{\pi}{4}\right] \\
&= \cos\left[6\left(\frac{n}{10}\right)^{5/6} x + \frac{5}{12}\left(\frac{n}{10}\right)^{1/2} x^3 + \frac{9}{64}\left(\frac{n}{10}\right)^{1/6} x^5 - \frac{n\pi}{2}\right] \\
&\quad \cdot \cos\left\{\left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6} x^7 - \frac{1}{2}\left(\frac{5}{32}\right)^{1/6} x\right] n^{-1/6}\right. \\
&\quad + \left[\frac{35}{576}\left(\frac{5}{32}\right)^{3/2} x^9 - \frac{1}{12}\left(\frac{5}{32}\right)^{1/2} x^3\right] n^{-1/2} \\
&\quad \left. + \left[\frac{189}{7040}\left(\frac{5}{32}\right)^{11/6} x^{11} - \frac{3}{80}\left(\frac{5}{32}\right)^{5/6} x^5\right] n^{-5/6}\right\} \\
&\quad + \sin\left[6\left(\frac{n}{10}\right)^{5/6} x + \frac{5}{12}\left(\frac{n}{10}\right)^{1/2} x^3 + \frac{9}{64}\left(\frac{n}{10}\right)^{1/6} x^5 - \frac{n\pi}{2}\right] \\
&\quad \cdot \sin\left\{\left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6} x^7 - \frac{1}{2}\left(\frac{5}{32}\right)^{1/6} x\right] n^{-1/6}\right. \\
&\quad + \left[\frac{35}{576}\left(\frac{5}{32}\right)^{3/2} x^9 - \frac{1}{12}\left(\frac{5}{32}\right)^{1/2} x^3\right] n^{-1/2} \\
&\quad \left. + \left[\frac{189}{7040}\left(\frac{5}{32}\right)^{11/6} x^{11} - \frac{3}{80}\left(\frac{5}{32}\right)^{5/6} x^5\right] n^{-5/6}\right\} + O(n^{-1}). \qquad (4.73)
\end{aligned}
$$

Let us rewrite this as

$$
\begin{aligned}
&\cos\left[\frac{n}{60}(60\theta - 15\sin 2\theta - 6\sin 4\theta - \sin 6\theta) + \frac{\theta}{2} - \frac{\pi}{4}\right] \\
&\quad = \cos F \cos G + \sin F \sin G + O(n^{-1}), \qquad (4.74)
\end{aligned}
$$

where F and G are the corresponding expressions in (4.73). Now, we will get the asymptotic expansions for $\cos G$ and $\sin G$. From (4.73), we see that

$$
\cos G = 1 - \frac{1}{2} G^2 + \frac{1}{24} G^4 + O\left(\frac{1}{n}\right) \qquad (4.75)
$$

and

$$
\sin G = G - \frac{1}{6} G^3 + \frac{1}{120} G^5 + O\left(\frac{1}{n}\right). \qquad (4.76)
$$

On the other hand, we have

$$G^2=\left\{\left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6}x^7-\frac{1}{2}\left(\frac{5}{32}\right)^{1/6}x\right]n^{-1/6}\right.$$
$$+\left[\frac{35}{576}\left(\frac{5}{32}\right)^{3/2}x^9-\frac{1}{12}\left(\frac{5}{32}\right)^{1/2}x^3\right]n^{-1/2}$$
$$\left.+\left[\frac{189}{7040}\left(\frac{5}{32}\right)^{11/6}x^{11}-\frac{3}{80}\left(\frac{5}{32}\right)^{5/6}x^5\right]n^{-5/6}\right\}^2$$
$$=\left[\frac{225}{3136}\left(\frac{5}{32}\right)^{7/3}x^{14}-\frac{15}{56}\left(\frac{5}{32}\right)^{4/3}x^8+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\right]n^{-1/3}$$
$$+2\left[\frac{75}{4608}\left(\frac{5}{32}\right)^{8/3}x^{16}-\frac{425}{8064}\left(\frac{5}{32}\right)^{5/3}x^{10}\right.$$
$$\left.+\frac{1}{24}\left(\frac{5}{32}\right)^{2/3}x^4\right]n^{-2/3}+O(n^{-1}), \tag{4.77}$$

$$G^3=\left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6}x^7-\frac{1}{2}\left(\frac{5}{32}\right)^{1/6}x\right]^3n^{-1/2}+3\left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6}x^7\right.$$
$$\left.-\frac{1}{2}\left(\frac{5}{32}\right)^{1/6}x\right]^2\cdot\left[\frac{35}{576}\left(\frac{5}{32}\right)^{3/2}x^9-\frac{1}{12}\left(\frac{5}{32}\right)^{1/2}x^3\right]n^{-5/6}+O(n^{-1})$$
$$=\left[\frac{3375}{175616}\left(\frac{5}{32}\right)^{7/2}x^{21}-\frac{675}{6272}\left(\frac{5}{32}\right)^{15/6}x^{15}+\frac{45}{224}\left(\frac{5}{32}\right)^{9/6}x^9\right.$$
$$\left.-\frac{1}{8}\left(\frac{5}{32}\right)^{1/2}x^3\right]n^{-1/2}+\left[\frac{23625}{1806336}\left(\frac{5}{32}\right)^{23/6}x^{23}\right.$$
$$-\left(\frac{1575}{32256}+\frac{675}{37632}\right)\left(\frac{32}{5}\right)^{17/6}x^{17}+\left(\frac{105}{2304}+\frac{45}{672}\right)\left(\frac{5}{32}\right)^{11/6}x^{11}$$
$$\left.-\frac{1}{16}\left(\frac{5}{32}\right)^{5/6}x^5\right]n^{-5/6}+O(n^{-1}), \tag{4.78}$$

$$G^4=\left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6}x^7-\frac{1}{2}\left(\frac{5}{32}\right)^{1/6}x\right]^4n^{-2/3}+O(n^{-1})$$
$$=\left[\frac{50625}{9834496}\left(\frac{5}{32}\right)^{14/3}x^{28}-\frac{13500}{351232}\left(\frac{5}{32}\right)^{22/6}x^{22}\right.$$
$$+\frac{1350}{12544}\left(\frac{5}{32}\right)^{16/6}x^{16}-\frac{60}{448}\left(\frac{5}{32}\right)^{10/6}x^{10}$$
$$\left.+\frac{1}{16}\left(\frac{5}{32}\right)^{4/6}x^4\right]n^{-2/3}+O(n^{-1}) \tag{4.79}$$

and

$$G^5=\left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6}x^7-\frac{1}{2}\left(\frac{5}{32}\right)^{1/6}x\right]^5 n^{-5/6}+O(n^{-1})$$

$$=\left[\frac{759375}{550731776}\left(\frac{5}{32}\right)^{35/6}x^{35}-\frac{253125}{19668992}\left(\frac{5}{32}\right)^{29/6}x^{29}\right.$$

$$+\frac{33750}{702464}\left(\frac{5}{32}\right)^{23/6}x^{23}-\frac{2250}{25088}\left(\frac{5}{32}\right)^{17/6}x^{17}$$

$$\left.+\frac{75}{896}\left(\frac{5}{32}\right)^{11/6}x^{11}-\frac{1}{32}\left(\frac{5}{32}\right)^{5/6}x^5\right]n^{-5/6}+O(n^{-1}). \tag{4.80}$$

Inserting (4.77) and (4.79) into (4.75), we obtain

$$\cos G=1+\left[\frac{-225}{6272}\left(\frac{5}{32}\right)^{7/3}x^{14}+\frac{15}{112}\left(\frac{5}{32}\right)^{4/3}x^8-\frac{1}{8}\left(\frac{5}{32}\right)^{1/3}x^2\right]n^{-1/3}$$

$$+\left[\frac{16875}{78675968}\cdot\left(\frac{5}{32}\right)^{14/3}x^{28}-\frac{4500}{2809856}\left(\frac{5}{32}\right)^{22/6}x^{22}\right.$$

$$+\left(\frac{-25}{1536}+\frac{225}{50176}\right)\left(\frac{5}{32}\right)^{8/3}x^{16}+\left(\frac{425}{8064}-\frac{5}{896}\right)\left(\frac{5}{32}\right)^{10/6}x^{10}$$

$$\left.+\left(-\frac{1}{24}+\frac{1}{384}\right)\left(\frac{5}{32}\right)^{2/3}x^4\right]n^{-2/3}+O(n^{-1}). \tag{4.81}$$

Similarly, inserting (4.78) and (4.80) into (4.76), we get

$$\sin G=\left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6}x^7-\frac{1}{2}\left(\frac{5}{32}\right)^{1/6}x\right]n^{-1/6}$$

$$+\left[\frac{-1125}{351232}\left(\frac{5}{32}\right)^{7/2}x^{21}+\frac{225}{12544}\left(\frac{5}{32}\right)^{15/6}x^{15}\right.$$

$$\left.+\left(\frac{35}{576}-\frac{15}{448}\right)\left(\frac{5}{32}\right)^{9/6}x^9-\frac{1}{16}\left(\frac{5}{32}\right)^{1/2}x^3\right]n^{-1/2}$$

$$+\left[\frac{50625}{4405854208}\left(\frac{5}{32}\right)^{35/6}x^{35}-\frac{16875}{157351936}\left(\frac{5}{32}\right)^{29/6}x^{29}\right.$$

$$+\left(\frac{6750}{16859136}-\frac{7875}{3612672}\right)\left(\frac{5}{32}\right)^{23/6}x^{23}$$

$$+\left(\frac{-225}{301056}+\frac{175}{21504}+\frac{75}{25088}\right)\left(\frac{5}{32}\right)^{17/6}x^{17}$$

$$+\left(\frac{15}{21504}+\frac{189}{7040}-\frac{105}{13824}-\frac{45}{4032}\right)\left(\frac{5}{32}\right)^{11/6}x^{11}$$

$$+\left(-\frac{1}{3840}+\frac{1}{96}-\frac{3}{80}\right)\left(\frac{5}{32}\right)^{5/6}x^5\Bigg]n^{-5/6}+O(n^{-1}). \qquad (4.82)$$

To simplify the computation, we rewrite $\cos G$ and $\sin G$ as follows:

$$\cos G = 1 + P_1(x)\,n^{-1/3} + P_2(x)\,n^{-2/3} + O(n^{-1}), \qquad (4.83)$$

$$\sin G = Q_1(x)\,n^{-1/6} + Q_2(x)\,n^{-1/2} + Q_3(x)\,n^{-5/6} + O(n^{-1}), \qquad (4.84)$$

where $P_i(x)$ and $Q_i(x)$ denote the corresponding polynomials in (4.81) and (4.82). Now, from (4.67), we have

$$\begin{aligned}(\sin\theta)^{-1/2} &= (1-\cos^2\theta)^{-1/4}\\ &= \left[1-\left(\frac{5}{32n}\right)^{1/3}x^2\right]^{-1/4}\\ &= 1+\frac{1}{4}\left(\frac{5}{32n}\right)^{1/3}x^2+\frac{5}{32}\left(\frac{5}{32n}\right)^{2/3}x^4+O(n^{-1}). \end{aligned}\qquad (4.85)$$

Combining (4.74), (4.83), (4.84) and (4.85), we obtain

$$(\sin\theta)^{-1/2}\cos\left[\frac{n}{60}(60\theta-15\sin 2\theta-6\sin 4\theta-\sin 6\theta)+\frac{\theta}{2}-\frac{\pi}{2}\right]$$

$$=\left[1+\frac{1}{4}\left(\frac{5}{32n}\right)^{1/3}x^2+\frac{5}{32}\left(\frac{5}{32n}\right)^{2/3}x^4+O(n^{-1})\right]$$

$$\cdot\{[1+P_1(x)n^{-1/3}+P_2(x)n^{-2/3}]\cdot\cos F$$

$$+[Q_1(x)n^{-1/6}+Q_2(x)n^{-1/2}+Q_3(x)n^{-5/6}]\sin F+O(n^{-1})\}$$

$$=\Bigg[1+P_1(x)n^{-1/3}+P_2(x)n^{-2/3}+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2$$

$$\cdot n^{-1/3}+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\cdot P_1(x)\cdot n^{-2/3}$$

$$+\frac{5}{32}\left(\frac{5}{32}\right)^{2/3}x^4\cdot n^{-2/3}\Bigg]\cos F$$

$$+\Bigg[Q_1(x)n^{-1/6}+Q_2(x)n^{-1/2}+Q_3(x)n^{-5/6}$$

$$+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\cdot Q_1(x)n^{-1/2}+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\cdot Q_2(x)n^{-5/6}$$

$$+\frac{5}{32}\left(\frac{5}{32}\right)^{2/3}x^4\cdot Q_1(x)n^{-5/6}\Bigg]\cdot\sin F+O(n^{-1})$$

$$=\left\{1+\left[P_1(x)+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\right]n^{-1/3}+\left[P_2(x)+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\cdot P_1(x)\right.\right.$$

$$\left.\left.+\frac{5}{32}\left(\frac{5}{32}\right)^{2/3}x^4\right]n^{-2/3}\right\}\cdot\cos F+\left\{Q_1(x)n^{-1/6}\right.$$

$$+\left[Q_2(x)+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\cdot Q_1(x)\right]n^{-1/2}$$

$$\left.+\left[Q_3(x)+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\cdot Q_2(x)+\frac{5}{32}\left(\frac{5}{32}\right)^{2/3}x^4\cdot Q_1(x)\right]n^{-5/6}\right\}$$

$$\cdot\sin F+O(n^{-1}), \tag{4.86}$$

where

$$P_1(x)+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2$$

$$=\left[\frac{-225}{6272}\left(\frac{5}{32}\right)^{7/3}x^{14}+\frac{15}{112}\left(\frac{5}{32}\right)^{4/3}x^8+\frac{1}{8}\left(\frac{5}{32}\right)^{1/3}x^2\right], \tag{4.87}$$

$$P_2(x)+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\cdot P_1(x)+\frac{5}{32}\left(\frac{5}{32}\right)^{2/3}x^4$$

$$=\left[\frac{16875}{78675968}\left(\frac{5}{32}\right)^{14/3}x^{28}-\frac{4500}{2809856}\left(\frac{5}{32}\right)^{22/6}x^{22}\right.$$

$$+\left(\frac{-75}{4608}+\frac{225}{50176}\right)\left(\frac{5}{32}\right)^{8/3}x^{16}+\left(\frac{425}{8064}-\frac{5}{896}\right)\left(\frac{5}{32}\right)^{10/6}x^{10}$$

$$\left.+\left(-\frac{1}{24}+\frac{1}{384}\right)\left(\frac{5}{32}\right)^{2/3}x^4\right]$$

$$+\frac{1}{4}\left(\frac{5}{32}\right)^{1/3}x^2\left[\frac{-225}{6272}\cdot\left(\frac{5}{32}\right)^{7/3}x^{14}\right.$$

$$\left.+\frac{15}{112}\left(\frac{5}{32}\right)^{4/3}x^8-\frac{1}{8}\left(\frac{5}{32}\right)^{1/3}x^2\right]+\frac{5}{32}\left(\frac{5}{32}\right)^{2/3}x^4$$

$$=\frac{16875}{78675968}\left(\frac{5}{32}\right)^{14/3}x^{28}-\frac{1125}{702464}\left(\frac{5}{32}\right)^{11/3}x^{22}$$

$$-\frac{3125}{150528}\left(\frac{5}{32}\right)^{8/3}x^{16}+\frac{325}{4032}\left(\frac{5}{32}\right)^{5/3}x^{10}+\frac{11}{128}\left(\frac{5}{32}\right)^{2/3}x^4, \tag{4.88}$$

$$
\begin{aligned}
Q_2(x) &+ \frac{1}{4}\left(\frac{5}{32}\right)^{1/3} x^2 \cdot Q_1(x) \\
&= \left[\frac{-1125}{351232}\left(\frac{5}{32}\right)^{7/2} x^{21} + \frac{225}{12544}\left(\frac{5}{32}\right)^{15/6} x^{15} \right. \\
&\quad \left. + \left(\frac{35}{576} - \frac{15}{448}\right) \cdot \left(\frac{5}{32}\right)^{9/6} \cdot x^9 - \frac{1}{16}\left(\frac{5}{32}\right)^{1/2} x^3 \right] \\
&\quad + \frac{1}{4}\left(\frac{5}{32}\right)^{1/3} x^2 \cdot \left[\frac{15}{56}\left(\frac{5}{32}\right)^{7/6} x^7 - \frac{1}{2}\left(\frac{5}{32}\right)^{1/6} x\right] \\
&= \frac{-1125}{351232}\left(\frac{5}{32}\right)^{7/2} x^{21} + \frac{225}{12544}\left(\frac{5}{32}\right)^{5/2} x^{15} \\
&\quad + \frac{95}{1008}\left(\frac{5}{32}\right)^{3/2} x^9 - \frac{3}{16}\left(\frac{5}{32}\right)^{1/2} x^3
\end{aligned} \tag{4.89}
$$

and finally,

$$
\begin{aligned}
Q_3(x) &+ \frac{1}{4}\left(\frac{5}{32}\right)^{1/3} x^2 \cdot Q_2(x) + \frac{5}{32}\left(\frac{5}{32}\right)^{2/3} x^4 \cdot Q_1(x) \\
&= \frac{50625}{4405854208}\left(\frac{5}{32}\right)^{35/6} x^{35} - \frac{16875}{157351936}\left(\frac{5}{32}\right)^{29/6} x^{29} \\
&\quad + \left(\frac{6750}{16859136} - \frac{7875}{3612672} - \frac{1125}{1404928}\right)\left(\frac{5}{32}\right)^{23/6} x^{23} \\
&\quad + \left(\frac{-225}{301056} + \frac{175}{21504} + \frac{75}{25088} + \frac{225}{50176}\right)\left(\frac{5}{32}\right)^{17/6} x^{17} \\
&\quad + \left(\frac{15}{21504} + \frac{189}{7040} - \frac{105}{13824} - \frac{45}{4032} + \frac{35}{2304} - \frac{15}{1792} \right. \\
&\quad \left. + \frac{75}{1792}\right)\left(\frac{5}{32}\right)^{11/6} x^{11} + \left(-\frac{1}{3840} + \frac{1}{96} - \frac{3}{80} - \frac{1}{64} - \frac{5}{64}\right)\left(\frac{5}{32}\right)^{5/6} x^5 \Big] \\
&= \frac{50625}{4405854208}\left(\frac{5}{32}\right)^{35/6} x^{35} - \frac{16875}{157351936}\left(\frac{5}{32}\right)^{29/6} x^{29} \\
&\quad - \frac{3625}{1404928}\left(\frac{5}{32}\right)^{23/6} x^{23} + \left(\frac{4475}{301056}\right)\left(\frac{5}{32}\right)^{17/6} x^{17} \\
&\quad + \frac{203881}{3548160}\left(\frac{5}{32}\right)^{11/6} x^{11} - \frac{31}{256}\left(\frac{5}{32}\right)^{5/6} x^5.
\end{aligned} \tag{4.90}
$$

Now, the theorem follows immediately from (4.86), (4.87), (4.88), (4.89), (4.90) and Theorem 1.

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